K-Theoretic and homological invariants of Dirac operators. Lecture 1.

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IMJ-PRG summer school 2025
Groupoids from a measurable, topological and geometric perspective

Outline of the first lecture

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Introduction

- We are interested in using K-theory and cyclic (co)homology of certain algebras in order to obtain numeric invariants of Dirac operators
- The main idea is to then use these invariants in order to prove geometric theorems.
- There is a hierarchy of geometric structures on which Dirac operators live: compact manifolds, fibrations of compact manifolds, Galois Γ-coverings, G-proper manifolds with G a Lie group, foliations
- Groupoids have a unifying role in this hierarchy
- still, we need to start from the basics !
- First of all, what is a Dirac operator ?

Dirac operators: preliminaries

- We consider a riemannian manifold (M, g) and a complex hermitian vector bundle $E \to M$ endowed with a ∇^E;
- we assume that for each $m \in M$ there exists a linear map $c_m : T_m^* M \otimes \mathbb{C} \to \mathrm{End(E_m)}$ satisfying :

$$c_m(\xi_m)c_m(\zeta_m) + c_m(\zeta_m)c_m(\xi_m) = -2g_m(\xi_m, \zeta_m)$$

- we assume that c_m depends smoothly on $m \in M$.
- using c_m, $m \in M$, we get $C^\infty(M, T^*M \otimes E) \xrightarrow{c} C^\infty(M, E)$ given by $c(\omega \otimes e)(m) := c_m(\omega(m))(e(m)) \in E_m$
- this is called a Clifford action on the sections of E

Dirac operators: definition

Definition

An operator of Dirac type is obtained taking the composition

$$C^\infty(M,E) \stackrel{\nabla^E}{\to} C^\infty(M, T^*M \otimes E) \stackrel{c}{\to} C^\infty(M,E).$$

Thus $D := c \circ \nabla^E$.

- if the Clifford action c is unitary, ∇^E is unitary and Clifford (i.e. compatible with the Levi-Civita connection) then D is formally self-adjoint , i.e. $D = D^*$
- if $\dim M = 2k$ then E is graded, $E = E^+ \oplus E^-$ and D is odd:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}. \quad D^- = (D^+)^*$$

Examples.

- The Gauss-Bonnet operator $d + d^*$ on differential forms, with $E = \Lambda^* X$.
  Here ∇^E is induced by Levi-Civita and
  $c_m(\xi_m) = \epsilon(\xi_m) - \iota(\xi_m)$ with $\epsilon(\xi_m)(\omega_m) = \xi_m \wedge \omega_m$ and
  $\iota(\xi_m)(\omega_m) =$ interior multiplication of ω_m by $\xi_m^* = g(\ ,\xi_m)$
- the Dolbeault operator on a complex hermitian manifold: $\overline{\partial} + \overline{\partial}^*$ with $E = \Lambda^{0,*}X$ and $c_m(\xi_m) = \epsilon(\xi_m^{0,1}) - \iota(\xi_m^{1,0})$,
- the spin-Dirac operator $D^{\mathrm{spin}} \equiv \not{D}$ on a spin manifold with $E = \not{S}$ the spinor bundle;
- the signature operator on an even dimensional orientable manifold D^{sign}: $D^{\mathrm{sign}} = d + d^*$ but with the grading $E = \Lambda^+ M \oplus \Lambda^- M$ defined in terms of Hodge-$\star$.

A crash-course on K-theory: $K^0(X)$

- If A is a commutative semigroup then the Grothendieck group associated to A is

$$G(A) := A \times A/\mathcal{R}$$

- $(x,y)\mathcal{R}(x',y')$ if $\exists\ z \in A$ such that $x + y' + z = x' + y + z$
- we think to $G(A)$ as formal differences of elements in A: $[(x,y)] \leftrightarrow x - y$
- $\mathbb{Z}$ =Grothendieck group associated to the semigroup $\mathbb{N}$
- consider $\mathrm{Vect}(X)$:= isomorphism classes of complex vector bundles over X; it is a semigroup !
- $K^0(X)$:= Grothendieck group associated to $\mathrm{Vect}(X)$

Crash-course: $K_0(C(X))$

- consider $V(C(X))$, the semigroup of isomorphism classes of finitely generated projective $C(X)$-modules
- recall: a finitely generated projective $C(X)$-module is, by definition, a direct summand of a free $C(X)$-module of finite rank ($C(X) \oplus \cdots \oplus C(X)$ (n times, for some n)).
- $K_0(C(X)) :=$ Grothendieck group associated to $V(C(X))$
- $K^0(X) \simeq K_0(C(X))$ (Swan' s theorem)
- there is in fact an isomorphism of semigroups: $\mathrm{Vect}(X) \xrightarrow{\varphi} V(C(X))$, $\varphi(E) := C(X, E)$.
- since a fin. gen. proj. $C(X)$-module S is a direct summand of $C(X) \oplus \cdots \oplus C(X)$ it follows that S is the image of a projector p_S in $M_{n\times n}(C(X))$
- in fact $V(C(X)) = \mathrm{Proj}(M_\infty(C(X)) =$ equivalence classes of projectors in $M_\infty(C(X))$ (equivalence relation $=$ similarity; we compare p in $M_{n\times n}$ and q in $M_{k\times k}$ by considering them in a bigger matrix algebra)

Crash-course: $K_*(A)$

- summarizing $K_0(C(X)) =$ Grothendieck group associated to the semigroup $\mathrm{Proj}(M_\infty(C(X))=$ formal differences of projectors in $M_\infty(C(X))$
- Let now A be a unital algebra
- $K_0(A) :=$ Grothendieck group associated to the semigroup $\mathrm{Proj}(M_\infty(A))$
- an element in $K_0(A)$ is a formal difference of projectors in $M_\infty(A)$
- let now A be, in addition, a C^*-algebra (or an algebra with "some topology")
- $K_1(A) := \mathrm{GL}_\infty(A)/\mathrm{GL}^0_\infty(A)$
- if $\psi : A \to B$ is a morphism then $\psi_* : K_*(A) \to K_*(B)$
- if A is not unital we define things through the unitalization A^+

Fundamental properties of $K_*(A)$

- Stability: if A is a C^*-algebra, $K_*(A) \simeq K_*(A \otimes \mathbb{K})$
- Suspension isomorphism: there exists a functorial isomorphism $\theta_A : K_1(A) \simeq K_0(S(A))$ with $S(A) := C_0(\mathbb{R}) \otimes A$
- Bott periodicity: there exists a functorial isomorphism $\beta_A : K_0(A) \simeq K_1(S(A))$
- if $\mathcal{A}$ is a dense Fréchet subalgebra of a C^*-algebra A which is holomorphically closed then $K_*(\mathcal{A}) = K_*(A)$
- if $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0$ is a short exact sequence of C^*-algebras then there exists a 6-terms periodic long exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/J) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(A/J) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[\iota_*]{} & K_1(J). \end{array} \tag{1}$$

A few examples

- $K^0(point) = \mathbb{Z}$
- $K^0(S^1) = \mathbb{Z}$; more generally $K^0(S^{2n+1}) = \mathbb{Z}$
- $K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$; more generally $K^0(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$
- $K_0(\mathbb{C}) = K_0(\mathbb{K}) = \mathbb{Z}$
- $K_1(\mathbb{C}) = K_1(\mathbb{K}) = 0$

Basic properties of Dirac operators.

- D is an elliptic differential operator: $\sigma_{\mathrm{pr}}(D)(\xi_m) = c_m(\xi_m)$ with $\xi_m \in T^*_m M$, and $c_m(\xi_m)$ is an isomorphism for $\xi_m \neq 0$
- D is essentially self-adjoint on $L^2(M, E)$; the unique self-adjoint closed extension has domain equal to $H^1(M, E)$, the first Sobolev space
- there exists a parametrix $Q : L^2(M, E) \to H^1(M, E)$ for D, i.e. an inverse modulo smoothing operators
- $Q \circ D = \mathrm{Id} - R$ and $D \circ Q = \mathrm{Id} - S$ with R and S smoothing (R and S are the remainders...)
- recall: R is a smoothing operator if $Ru(x) = \int_M k_R(x, y) u(y) dy$ with $K_R \in C^\infty(M \times M, E \boxtimes E^*)$
- if M is compact without boundary then a smoothing operator R defines a compact operator on L^2 and on each Sobolev H^k.
- in fact R defines a trace class operator; moreover $\mathrm{Tr}(R) = \int_M \mathrm{tr}_x(K_R(x, x)) dx$

Basic properties of Dirac operators (cont)

- Summarizing: D is invertible modulo compacts
- thus (Atkinson's theorem) D is Fredholm: $\dim \mathrm{Ker}(D) < \infty$ and $\dim \mathrm{coker}(D) < \infty$
- $\mathrm{ind}(D) := \dim \mathrm{Ker}(D) - \dim \mathrm{coker}(D) = \dim \mathrm{Ker}(D) - \dim \mathrm{Ker}(D^*)$
- if $\dim M = 2k$ then E is graded, $E = E^+ \oplus E^-$ and D is odd:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}. \quad D^- = (D^+)^*$$

- $\mathrm{ind}(D) = 0$ (since $D = D^*$) but if $\dim M = 2k$, $\mathrm{ind}\, D^+ \neq 0$
- Calderon's formula: $\mathrm{ind}(D^+) = \mathrm{Tr}(S_+^N) - \mathrm{Tr}(S_-^N)\ \forall N \geq 1$. Here $S_\pm$ are the remainders of a parametrix for D^+

A glimpse of pseudodifferential operators

- The parametrix Q, i.e. the pseudo-inverse of D, has a special structure: it is a pseudodifferential operator of order -1: $Q \in \Psi^{-1}(M, E)$.
- let U be an open ball in $\mathbb{R}^n$
- $\Psi^\ell(U)$ is the space of linear operators $P : C_c^\infty(U) \to C^\infty(U)$ that can be written as

$$(Pu)(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\widehat{u}(\xi) d\xi$$

- $p \in C^\infty(U \times \mathbb{R}^n)$ is a function of compact x-support uniformly in ξ satisfying the following: $\forall \alpha, \beta \; \exists \, C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta (p(x,\xi))| < C_{\alpha,\beta}(1 + |\xi|)^{\ell - |\beta|}$$

- we have just defined the space of symbols of order ℓ: $S^\ell(U \times \mathbb{R}^n)$
- we define $\Psi^\ell(U, \mathbb{C}^k)$ in terms of matrices of such operators

A glimpse of pseudodifferential operators (cont)

- we define $\Psi^\ell(M,E)$ by globalizing this local definition
- composition formula: $\Psi^\ell(M,E) \circ \Psi^k(M,E) \subset \Psi^{\ell+k}(M,E)$
- $\cap_{k\in\mathbb{Z}}\Psi^k(M,E) := \Psi^{-\infty}(M,E) =$ smoothing operators
- the local symbols give a well defined principal symbol $\sigma_{\mathrm{pr}}(P)(\xi_m) : E_m \to E_m \ \forall m \in M$
- in fact $\sigma_{\mathrm{pr}}(P) \in C^\infty(T^*M, \mathrm{End}(\pi^*E, \pi^*E))$
- if $P, Q \in \Psi^\ell(M,E)$ and $\sigma_{\mathrm{pr}}(P) = \sigma_{\mathrm{pr}}(Q)$ then $P - Q \in \Psi^{\ell-1}(M,E)$
- the parametrix of D is obtained from an inductive procedure; the first step is to take the operator in $\Psi^{-1}(M,E)$ with symbol given by the inverse of the symbol of D (which is Clifford multiplication).
- the inverse symbol is well defined because D is elliptic

A new look at the Fredholm index

- we are on an even dimensional compact manifold M and a $\mathbb{Z}_2$-graded odd Dirac operator D acting on the sections of $E = E^+ \oplus E^-$
- I want to give a different description of $\mathrm{ind}(D^+) \in \mathbb{Z}$
- Claim: there exists an index class $\mathrm{Ind}(D) \in K_0(\Psi^{-\infty}(M, E))$ and a trace functional $\tau : \Psi^{-\infty}(X, E) \to \mathbb{C}$ such that
$$\mathrm{ind}(D^+) = \tau(\mathrm{Ind}(D))$$
- here τ is extended to matrices in the obvious way: $\tau(p_{ij}) := \sum_j \tau(p_{jj})$

The index class $\text{Ind}(D) \in K_0(\Psi^{-\infty}(X, E))$

- Let $Q \in \Psi^{-1}(M, E^-, E^+)$ be a parametrix for D^+ with remainders $S_\pm \in \Psi^{-\infty}(M, E^\pm)$
- Consider the 2×2 matrix

$$P := \begin{pmatrix} S_+^2 & S_+(I + S_+)Q \\ S_- D^+ & I - S_-^2 \end{pmatrix}.$$

- Entries are in the unitalization of $\Psi^{-\infty}(M, E)$
- It is a projector
- by definition $\text{Ind}(D) := [P] - [e_1] \in K_0(\Psi^{-\infty}(M, E))$
- here $e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1_{E^-} \end{pmatrix}$ (also a projector)
- Conclusion: we have defined $\text{Ind}(D) \in K_0(\Psi^{-\infty}(M, E))$

From the index class $\mathrm{Ind}(D)$ to $\mathrm{ind}(D^+)$

- define $\tau : \Psi^{-\infty}(M, E) \to \mathbb{C}$ as $\tau(R) := \mathrm{Tr}(R)$
- we know (Calderon) that $\mathrm{ind}(D^+) = \mathrm{Tr}(S_+^N) - \mathrm{Tr}(S_-^N)$, $N \geq 1$
- it is now clear that $\tau(\mathrm{Ind}(D)) = \mathrm{ind}(D^+)$ since

$$\mathrm{Tr}\begin{pmatrix} S_+^2 & S_+(I + S_+)Q \\ S_- D^+ & -S_-^2 \end{pmatrix} = \mathrm{Tr}(S_+^2) - \mathrm{Tr}(S_-^2) = \mathrm{ind}(D^+)$$

by Calderon formula

0-cyclic cocycles

- let $\mathcal{A}$ be a Fréchet algebra
- $HC^0(\mathcal{A}) = \{\tau : \mathcal{A} \to \mathbb{C} \text{ continuous } \mid \tau(a_0 a_1) = \tau(a_1 a_0)\}$
- thus $HC^0(\mathcal{A})$ = continuous traces on $\mathcal{A}$
- $HC^0(\mathcal{A})$ is the 0-degree cyclic cohomology group associated to $\mathcal{A}$
- we have a pairing $\langle \cdot, \cdot \rangle : K_0(\mathcal{A}) \otimes HC^0(\mathcal{A}) \longrightarrow \mathbb{C}$: $\langle [(p_{ij})], \tau \rangle := \sum \tau(p_{jj})$
- our example: $\mathcal{A} = \Psi^{-\infty}(X, E)$ and $\tau \in HC^0(\Psi^{-\infty}(X, E))$ given by the trace: $\Psi^{-\infty}(X, E) \ni R \to \tau(R) := \mathrm{Tr}(R) \in \mathbb{C}$;
- denote $\tau = \mathsf{Tr}$;
- we have proved that $\langle \mathsf{Ind}(D), \mathrm{Tr} \rangle = \mathsf{ind}(D^+)$
- so, we have expressed $\mathsf{ind}(D^+)$ as a pairing between an index class $\mathsf{Ind}(D)$ and a 0-degree cyclic cocycle Tr.

Comments

- is all this really interesting ?
- yes and no
- recall that if $\mathcal{A}$ is a dense Fréchet subalgebra of a C^*-algebra A which is holomorphically closed then $K_*(\mathcal{A}) = K_*(A)$
- Example: $\Psi^{-\infty}(M)$ is dense and holomorphically closed in $\mathbb{K}(L^2)$. So $K_*(\Psi^{-\infty}(M)) = K_*(\mathbb{K}(L^2))$
- but $K_0(\mathbb{K}(L^2)) = \mathbb{Z}$ and $K_1(\mathbb{K}(L^2)) = 0$
- so, no, nothing really new.......
- on the other hand, yes, the point of view of going to K-theory and cyclic cohomology is VERY important
- to get something new we shall need to pass to the universal cover $\widetilde{M}$ where we shall get interesting K-theory groups

More on the index class

- where does the definition of the index class $\mathrm{Ind}(D) = [P] - [e_1]$ come from ?!
- we now expunge E from the notation.....
- recall that if $\mathcal{A}$ is a dense Fréchet subalgebra of a C^*-algebra A which is holomorphically closed then $K_*(\mathcal{A}) = K_*(A)$
- E.g.: we have seen that $\Psi^{-\infty}(M)$ is dense holomorphically closed in $\mathbb{K}(L^2)$. So $K_*(\Psi^{-\infty}(M)) = K_*(\mathbb{K}(L^2))$
- from the properties of the principal symbol we have $0 \to \Psi^{-1}(M) \to \Psi^0(X) \xrightarrow{\sigma} C^\infty(S^*M) \to 0$
- consider its C^* closure in $\mathcal{B}(L^2)$
$$0 \to \mathbb{K}(L^2) \to \overline{\Psi^0(M)} \xrightarrow{\sigma} C(S^*M) \to 0$$
- get a long exact sequence in K-theory $\cdots \to K_1(C(S^*X)) \xrightarrow{\partial} K_0(\mathbb{K}(L^2)) \to \cdots$.
- an elliptic operator P defines a class $\sigma_P \in K_1(C(S^*M))$
- unraveling the definition one discovers that the index class we have introduced is $\partial(\sigma_P) \in K_0(\mathbb{K}(L^2)) = K_0(\Psi^{-\infty}(M))$