K-Theoretic and homological invariants of Dirac operators

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Groupoids from a measurable, topological and geometric perspective

Outline of the second lecture

Summary of the first lecture

Index classes on Galois coverings

The analytic surgery sequence

Summary

- we have defined Dirac operators and explained that they are Fredholm
- ind D^+ = dim Ker D^+ − dim Ker D^-
- we talked briefly about the filtered algebra of pseudodifferential operators $\{\Psi^k(M,E)), k \in \mathbb{Z}\}$
- $\cap_{k\in\mathbb{Z}}\Psi^k(M,E)) =: \Psi^{-\infty}(M,E)$ is the algebra of smoothing operators
- We have defined $K_*(A)$, with A a C^*-algebra and we have stated the main properties
- we have expressed ind(D^+) as a pairing between an index class Ind$(D)\in K_0(\Psi^{-\infty}(M,E))$ and a 0-degree cyclic cocycle Tr $\in HC_0(\Psi^{-\infty}(M,E))$
- $\langle \mathrm{Ind}(D), \mathrm{Tr}\rangle = \mathrm{ind}(D^+)$
- we gave two descriptions of Ind$(D)\in K_0(\Psi^{-\infty}(M,E))$

Operators on $\widetilde{X}$

- we now pass to the universal cover $\widetilde{X}$ or to any Galois Γ-cover
- we lift all the data defining our Dirac operator to $\widetilde{X}$ and get a Γ-equivariant Dirac operator $\widetilde{D}$
- we consider $\Psi^{-\infty}_{\Gamma,c}(\widetilde{X})$, the Γ-equivariant smoothing operators of Γ-compact support
- this means that the support of the Schwartz kernel is compact when projected to $\widetilde{X} \times \widetilde{X}/\Gamma$ (Γ acting diagonally)
- given $\widetilde{D}$ ($\mathbb{Z}_2$-graded odd) we can construct a parametrix $\widetilde{Q} : L^2(\widetilde{X}, \widetilde{E}^-) \to \mathrm{Dom}(\widetilde{D}^+)$ for $\widetilde{D}^+$ with remainders $\widetilde{S}_\pm \in \Psi^{-\infty}_{\Gamma,c}(\widetilde{X}, \widetilde{E}^\pm)$:
  $Q \circ \widetilde{D}^+ = \mathrm{Id} - \widetilde{S}_+, \quad \widetilde{D}^+ \circ Q = \mathrm{Id} - \widetilde{S}_-$

Index classes

- Consider now the projector

$$\widetilde{P} := \begin{pmatrix} \widetilde{S}_+^2 & \widetilde{S}_+(I + \widetilde{S}_+)\widetilde{Q} \\ \widetilde{S}_- \widetilde{D}^+ & I - \widetilde{S}_-^2 \end{pmatrix}.$$

- Entries are in the unitalization of $\Psi^{-\infty}_{\Gamma,c}(\widetilde{X}, \widetilde{E})$
- By definition $\mathrm{Ind}_{\Gamma,c}(\widetilde{D}) := [\widetilde{P}] - [e_1]$ in $K_0(\Psi^{-\infty}_{\Gamma,c}(\widetilde{X}, \widetilde{E}))$
- this is the compactly supported index class
- define $C^*(\widetilde{X}, \widetilde{E})^\Gamma := \overline{\Psi^{-\infty}_{\Gamma,c}(\widetilde{X}, \widetilde{E})}$, the C^*-closure in $\mathcal{B}(L^2)$
- $C^*(\widetilde{X}, \widetilde{E})^\Gamma$ is known as the Roe algebra of the pair $(\widetilde{X}, \widetilde{E})$
- the same formula defines $\mathrm{Ind}_\Gamma(\widetilde{D}) \in K_0(C^*(\widetilde{X}, \widetilde{E})^\Gamma)$
- this is the C^*-index class and it is the index class we are actually interested in (will explain why ...)

Index classes (cont)

- one can prove that $K_*(C^*(\widetilde{X}, \widetilde{E})^\Gamma) = K_*(C_r^*\Gamma)$ and the latter is a very interesting K-theory group (in contrast with $K_*(\mathbb{K}(L^2))$!!)
- recall that $C_r^*\Gamma$ is the closure of $\mathbb{C}\Gamma$ in $\mathcal{B}(\ell^2(\Gamma))$, where $g \in \mathbb{C}\Gamma \equiv \{f : \Gamma \to \mathbb{C}$ of compact support$\}$ acts by convolution on $\ell^2(\Gamma)$
- it is customary to write elements in $\mathbb{C}\Gamma$ as finite sums $\sum \alpha_\gamma \gamma$, with $\alpha_\gamma \in \mathbb{C}$

Exact sequences on $\widetilde{X}$ (cont)

- the Γ-index class can be framed as in the closed case
- there exists a short exact sequence
  $0 \to C^*(\widetilde{X}, \widetilde{E})^\Gamma \to \Psi^0_\Gamma(\widetilde{X}, \widetilde{E}) \xrightarrow{\sigma} C(S^*X, \mathrm{End}(\pi^* E)) \to 0$
  where $\pi : S^*X \to X$
- there exists a long exact sequence in K-theory:
  $\cdots K_1(C(S^*X), \mathrm{End}(\pi^* E)) \xrightarrow{\delta} K_0(C^*(\widetilde{X}, \widetilde{E}^\Gamma) \cdots$
- our C^*-index class $\mathrm{Ind}_\Gamma(\widetilde{D}) \in K_0(C^*(\widetilde{X}, \widetilde{E}^\Gamma)$ is the image under δ of the symbol class, exactly as before
- Conclusion: we have defined an index class in a very rich K-theory group !

Atiyah Γ-index

- on the algebra of Γ-equivariant smoothing operators we have Atiyah's Γ-trace
- $\mathrm{Tr}_\Gamma(R) = \int_{\mathcal{F}} \mathrm{tr}_m K_R(m,m) dm$ with $\mathcal{F}$ a fundamental domain
- we can define $\mathrm{ind}_\Gamma(\widetilde{D}^+) = \mathrm{Tr}_\Gamma(\Pi_{\mathrm{Ker}_+}) - \mathrm{Tr}_\Gamma(\Pi_{\mathrm{Ker}_-})$ where $\Pi_{\mathrm{Ker}_\pm}$ are the projections onto the kernels of $\widetilde{D}^\pm$
- Atiyah's L^2-index theorem: $\mathrm{ind}_\Gamma(\widetilde{D}^+) = \mathrm{ind}(D^+)$
- as before, the Γ-index of Atiyah, $\mathrm{ind}_\Gamma(\widetilde{D}^+)$, is the pairing of $\mathrm{Ind}_{\Gamma,c}(\widetilde{D}) \in K_0(\Psi^{-\infty}_{\Gamma,c}(\widetilde{M}, \widetilde{E}))$ with the Γ-Trace, an element in $HC^0(\Psi^{-\infty}_{\Gamma,c}(\widetilde{M}))$
- viz. : $\mathrm{ind}_\Gamma(\widetilde{D}^+) = \langle \mathrm{Ind}_{\Gamma,c}(\widetilde{D}), \mathrm{Tr}_\Gamma \rangle$
- here we are thus using the pairing
$$\langle\, , \rangle : K_*(\Psi^{-\infty}_{\Gamma,c}(\widetilde{M})) \otimes HC^0(\Psi^{-\infty}_{\Gamma,c}(\widetilde{M})) \to \mathbb{C}$$

A crash-course on cyclic cohomology

- $\mathcal{A}$ = Fréchet algebra over $\mathbb{C}$
- Hochschild cochains of degree k: $C^k(\mathcal{A})$
- $C^k(\mathcal{A})$: all continuous $k+1$-linear functionals on $\mathcal{A}$
- Hochschild codifferential b: $C^k(\mathcal{A}) \to C^{k+1}(\mathcal{A})$

$$
\begin{aligned}
& b\Phi(a_0 \otimes \cdots \otimes a_{k+1}) \\
= & \sum_{i=0}^{k} (-1)^i \Phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) \\
+ & (-1)^{k+1} \Phi(a_{k+1} a_0 \otimes a_1 \otimes \cdots \otimes a_k).
\end{aligned}
$$

- Hochschild cohomology of $\mathcal{A}$ is cohomology of $(C^*(\mathcal{A}), b)$
- a Hochschild k-cochain $\Phi \in C^k(\mathcal{A})$ is called *cyclic* if $\Phi(a_k, a_0, \ldots, a_{k-1}) = (-1)^k \Phi(a_0, a_1, \ldots, a_k)$
- $C^k_\lambda(\mathcal{A})$= {cyclic cochains} ; it is closed under b.
- cyclic cohomology $HC^*(A)$ = cohomology of $(C^k_\lambda(\mathcal{A}), b)$.

Pairing $K_0(\mathcal{A})$ with $HC^{\mathrm{evev}}((\mathcal{A})$

- recall
  $HC^0(\mathcal{A}) = \{\Phi : \mathcal{A} \to \mathbb{C} \text{ continuous } \mid \Phi(a_0 a_1) = \Phi(a_1 a_0)\}$
- so that $HC^0(\mathcal{A}) =$ continuous traces on $\mathcal{A}$
- we have seen that $\langle \cdot, \cdot \rangle : K_0(\mathcal{A}) \otimes HC^0(\mathcal{A}) \longrightarrow \mathbb{C}$:
  $\langle (p_{ij}), \Phi \rangle := \sum \Phi(p_{jj})$
- more generally $\langle \cdot, \cdot \rangle : K_0(\mathcal{A}) \otimes HC^{2\mathrm{k}}(\mathcal{A}) \longrightarrow \mathbb{C}$

$$\langle [P], \Phi \rangle = \frac{1}{k!} \sum_{i_0, i_1, \ldots i_{2k}} \Phi(p_{i_0 i_1}, \ldots p_{i_{2k} i_0})$$

- in particular we have a pairing
  $K_0(\Psi^{-\infty}_{\Gamma,c}(\widetilde{M})) \otimes HC^{\mathrm{even}}(\Psi^{-\infty}_{\Gamma,c}(\widetilde{X})) \to \mathbb{C}$
- this pairing can be used in order to extract numbers out of our index class: these are the higher indices

Higher indices (Connes and Moscovici)

- we assume that X is even dimensional
- consider $\alpha \in H^k(\Gamma)$
- recall: the chains are antisymmetric left-invariant functions $\alpha : \Gamma^{k+1} \to \mathbb{C}$
- and the differential is $\delta\alpha(g_0, g_1, \dots, g_{k+1}) = \sum_{j=1}^{k+1}(-1)^j \alpha(g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_k)$
- classic: $H^k(\Gamma)$ is isomorphic to $H^k(B\Gamma)$
- consider $HC^*(\mathbb{C}\Gamma)$
- a cycle in $HC^k(\mathbb{C}\Gamma)$ is a (k+1)-multilinear functional τ on $\mathbb{C}\Gamma$; we write it like $\tau(g_0, g_1, \dots, g_k)$
- we can define a cyclic class $[\tau_\varphi^\Gamma] \in HC^k(\mathbb{C}\Gamma)$ given by the cyclic cocycle: $\tau_\alpha^\Gamma(g_0, g_1, \dots, g_k) = 0$ if $g_0 \cdots g_k \neq e$
  $\tau_\alpha^\Gamma(g_0, g_1, \dots, g_k) = \alpha(g_0, g_0 g_1, \dots, g_0 \cdots g_k)$ if $g_0 \cdots g_k = e$

Connes and Moscovici (cont)

- we can also define a cyclic class $\tau_{[\alpha]} \in HC^k(\Psi^{-\infty}_{\Gamma,c}(\widetilde{X}))$
- by definition $\tau_{[\alpha]}(A_0,\dots,A_k) = \sum_{g_0 \dots g_k = 1} \tau^{\Gamma}_{\alpha}(g_0, g_1, \dots, g_k)$ $\int \chi(x_0)A_0(x_0, g_0 \cdot x_1) \cdots \chi(x_k)A_k(x_k, g_k \cdot x_0)dx_0 \cdots dx_k$.
- here the integration is over $\widetilde{X}^{k+1}$
- χ is a cut-off function satisfying $\sum_{\gamma \in \Gamma} \chi(\gamma^{-1} \cdot x) = 1, \qquad \text{for all } x \in \widetilde{X}$
- thus $\tau_{[\alpha]}$ defines a homom. $\langle \cdot, \tau_{[\alpha]} \rangle : K_0(\Psi^{-\infty}_{\Gamma,c}(\widetilde{X})) \to \mathbb{C}$
- applied to $\mathrm{Ind}_{\Gamma,c}(\widetilde{D})$ this defines a number, which is a higher (compactly supported) index
- Connes and Moscovici prove a geometric formula for these higher indices (à la Atiyah-Singer).
- the right hand side of these formulae are very important geometric objects (higher signatures, higher $\widehat{A}$-genera)

Connes and Moscovici (cont)

- in general it is not known if $\langle \cdot, \tau_{[\alpha]} \rangle$ extends to $K_0(C^*(\widetilde{X})^\Gamma)$
- Connes-Moscovici: if Γ is Gromov hyperbolic or of polynomial growth then there exists a dense holomorphically closed subalgebra $\Psi^{-\infty}_{\Gamma,c}(\widetilde{X}) \subset \mathcal{B}(\widetilde{X})^\Gamma \subset C^*(\widetilde{X})^\Gamma$ such that $\langle \cdot, \tau_{[\alpha]} \rangle$ extends to $K_*(\mathcal{B}(\widetilde{X})^\Gamma) = K_*(C^*(\widetilde{X})^\Gamma)$.
- using the C^*-index class we obtain under these assumptions higher C^*-indices $\langle \mathrm{Ind}_\Gamma(\widetilde{D}), \tau_{[\alpha]} \rangle$
- there are cases in which Atiyah's Γ-index $\mathrm{ind}_\Gamma(\widetilde{D}^+)$ vanishes but a higher index does not, for example if X is equal to the n-torus

Invertible operators

- if $\widetilde{D}$ is L^2-invertible then the index class $\mathrm{Ind}_\Gamma(\widetilde{D}) \in K_*(C^*(\widetilde{X})^\Gamma)$ vanishes
- for example, if M is spin and g is of positive scalar curvature, then $\mathrm{Ind}_\Gamma(\widetilde{D}^{\mathrm{spin}}) = 0$ (it is here that we need C^*-algebras !)
- to prove this vanishing we use: $(\widetilde{D}^{\mathrm{spin}})^2 = \nabla\nabla^* + \mathrm{scal}_{\widetilde{g}}/4$ (Lichnerowicz formula)
- if we want to distinguish metrics of positive scalar curvature (up to isotopy, for example) we need finer invariants
- similarly if X is the union of two homotopy equivalent orientable manifolds then $\mathrm{Ind}_\Gamma(\widetilde{D}^{\mathrm{sign}}) = 0$ (Kasparov, Hilsum-Skandalis)
- if we want to distinguish non-diffeomorphic manifolds that are homotopy equivalent we need once again finer invariants

Secondary numeric invariant

- we have the Cheeger-Gromov rho invariant:
  $\rho_{\mathrm{CG}}(\widetilde{D}) := \eta_{(2)}(\widetilde{D}) - \eta(D)$
- here the L^2-eta invariant and the eta invariant are defined by :

$$\eta_{(2)}(\widetilde{D}) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_\Gamma(\widetilde{D} \exp(-(t\widetilde{D})^2) dt$$

$$\eta(D) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}(D \exp(-(tD)^2) dt$$

Secondary numeric invariants; some geometric applications

These two invariants are very useful. Here are two results.

Theorem

(Chang-Weinberger '03) If M is a compact oriented manifold of dimension $4k+3$, $k>0$, such that $\pi_1(M)=\Gamma$ has torsion, then there are infinitely many manifolds that are homotopic equivalent to M but not diffeomorphic to it. They are distinguished by the Cheeger-Gromov rho-invariant of the signature operator.

Theorem

(P.-Schick '07) M is spin of dimension $4k+3$, $k>0$, with g of PSC and $\Gamma=\pi_1(M)$ with torsion.
Then $|\pi_0(\mathcal{R}^+(M)/\mathrm{Diffeo}(M))|=\infty$ and the connected components are distinguished by the Cheeger-Gromov rho invariant of the spin-Dirac operator.

Basic questions and very short answers

- let $\widetilde{D}$ be L^2-invertible
- Q1: is there a rho class $\rho(\widetilde{D})$ in a K-theory group producing this secondary numeric invariant upon the use of a suitable trace ?
- Q2: can we extract higher rho numbers out of this rho class by pairing it with higher cyclic cocycles ?
- Q3: can we use these higher rho numbers in order to study the moduli space of positive scalar curvature metrics ?
- A1: yes, there is a rho class in the analytic surgery sequence of Higson and Roe
- A2: yes, under additional assumptions on the group Γ we can define higher rho numbers (main result of P-Schick-Zenobi)
- A3: yes, we can apply these invariants in order to study moduli spaces of positive scalar curvature metrics (P-Shick-Zenobi).

The Higson-Roe surgery sequence: preliminaries

- let (X, d) be a metric space
- we are interested in $C_0(X)$-modules, that is Hilbert spaces H on which elements of $C_0(X)$ act as bounded operators
- a bounded operator T on H has propagation $\leq R$ if $\psi \circ T \circ \phi \equiv 0$ whenever the distance between the supports of ψ and ϕ is $\geq R$
- Let now X be a smooth riemannian manifold
- Example 1: it is a classic result that e^{itD} has propagation $\leq |t|$ if D is a Dirac operator; this will be important
- Example 2: a smoothing operator with support in an R-neighbourhood of the diagonal has propagation $\leq R$
- consider $D^*_c(\widetilde{X})^\Gamma :=$ Γ-equivariant bounded operators on $L^2(\widetilde{X})$ of finite propagation and pseudolocal (i.e. $[f, T]$ is compact $\forall f \in C^\infty_c(\widetilde{X})$);it's a subalgebra of $\mathcal{B}(L^2)$

The Higson-Roe surgery sequence

- $C^*_c(\widetilde{X})^\Gamma \subset \mathcal{B}(L^2(\widetilde{X}))$ is the subalgebra of $D^*_c(\widetilde{X})^\Gamma$ made of operators that are, in addition, locally compact (i.e. fT is a compact operator for any $f \in C^\infty_c(\widetilde{X})$).
- $C^*_c(\widetilde{X})^\Gamma$ is an ideal in $D^*_c(\widetilde{X})^\Gamma$
- $D^*(\widetilde{X})^\Gamma := \overline{D^*_c(\widetilde{X})^\Gamma}^{\mathcal{B}(L^2)}$
- $C^*(\widetilde{X})^\Gamma := \overline{C^*_c(\widetilde{X})\Gamma}^{\mathcal{B}(L^2)}$; this is the original definition of Roe algebra (it is compatible with our old definition)
- the Roe algebra $C^*(\widetilde{X})^\Gamma$ is an ideal in $D^*(\widetilde{X})^\Gamma$
- get $0 \to C^*(\widetilde{X})^\Gamma \to D^*(\widetilde{X})^\Gamma \to D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma \to 0$
- get $\cdots \to K_*(D^*(\widetilde{X})^\Gamma) \to K_*(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma) \xrightarrow{\delta} K_{*+1}(C^*(\widetilde{X})^\Gamma) \to \cdots$
- this is the analytic surgery sequence of Higson and Roe

More on the Higson-Roe surgery sequence

Important facts:

- we know already that $K_*(C^*(\widetilde{X})^\Gamma) = K_*(C_r^*\Gamma)$
- Paschke duality:
  $K_*(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma) = K^\Gamma_{*+1}(\widetilde{X}) \equiv K_{*+1}(X)$
- here the K-**homology** groups $K_*(X)$ have appeared; we shall see the definition at some point, but not now
- $\mathrm{S}^\Gamma_*(\widetilde{M}) := K_{*+1}(D^*(\widetilde{M})^\Gamma)$ is the analytic structure group
- We can rewrite the surgery sequence as

$$\cdots \to K_{*+1}(C_r^*\Gamma) \to \mathrm{S}^\Gamma_*(\widetilde{X}) \to K_*(X) \xrightarrow{\delta} K_*(C_r^*\Gamma) \to \cdots$$

- these groups behave functorially. So, if $\widetilde{u} : \widetilde{X} \to E\Gamma$ is a Γ-equiv. classifying map then we can use $\widetilde{u}_*$ to map to the universal HR sequence:

  $$\cdots \to K_{*+1}(C_r^*\Gamma) \to \mathrm{S}^\Gamma_*(E\Gamma) \to K_*(B\Gamma) \xrightarrow{\delta} K_*(C_r^*\Gamma) \to \cdots$$

  The homomorphism δ is one of the many incarnations of the assembly map.

Index classes and Rho-classes

- as usual I shall not write the bundle E in the notation.....
- we fix a chopping function χ: a smooth odd function $\to \pm 1$ as $x \to \pm\infty$. Notice that $\chi^2 - 1 \in C_0(\mathbb{R})$.
- from the finite propagation property of $\exp(iDt)$ we have $\chi(D) \in D^*(\widetilde{X})^\Gamma$ and $\phi(D) \in C^*(\widetilde{X})^\Gamma$ if $\phi \in C_0(\mathbb{R})$.
- So $\chi(D)$ is an involution in $D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma$
- let $n := \dim M$ be even; then $E = E^+ \oplus E^-$
- $[D]$:= $[U^* \chi(\widetilde{D})_+] \in K_1(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma) = K_0(X)$, with U a suitable (local) unitary operator $L^2(M, E^+) \to L^2(M, E^-)$.
- if n is odd $[D]$:= $[\frac{1}{2}(1 + \chi(\widetilde{D})] \in K_0(D^*(\widetilde{X})^\Gamma / C^*(\widetilde{X})^\Gamma) =$$K_1(X)$.
- $[D] \in K_*(X)$ is the fundamental class associated to D
- recall δ in the Higson-Roe surgery sequence:
$$\cdots \to K_{*+1}(C^*_r \Gamma) \to \mathrm{S}^\Gamma_*(\widetilde{X}) \to K_*(X) \xrightarrow{\delta} K_*(C^*_r \Gamma) \to \cdots$$
- $\mathrm{Ind}_{\mathrm{Roe}}(\widetilde{D}) := \delta[D] \in K_n(C^*(\widetilde{X})^\Gamma)$ is the Roe index class.
- Important: $\mathrm{Ind}_{\mathrm{Roe}}(\widetilde{D}) = \mathrm{Ind}_\Gamma(\widetilde{D})$ in $K_*(C^*(\widetilde{X}))$, our friend !

Rho-classes

- Assume that $\widetilde{D}$ is L^2-invertible.
- $\widetilde{D}$ L^2-invertible $\Rightarrow$ $\mathrm{Ind}_\Gamma(\widetilde{D}) = 0$ in $K_*(C^*(\widetilde{X})^\Gamma) = K_*(C^*_r\Gamma)$
- thus $\delta[D] \equiv \mathrm{Ind}_\Gamma(\widetilde{D}) = 0$
- $\rho(\widetilde{D}) \in \mathrm{S}^\Gamma_*(\widetilde{X})$ is a natural **lift** of $[D]$.

$$\mathrm{S}^\Gamma_*(\widetilde{X}) \longrightarrow K_*(X) \xrightarrow{\delta} K_*(C^*_r\Gamma)$$

$$\rho(\widetilde{D}) \qquad\qquad [D] \xrightarrow{\quad\delta\quad} 0$$

Rho-classes (cont)

- More precisely: if $\widetilde{D}$ is L^2-invertible and $\chi = 1$ on the positive part of the spectrum, then the rho classes in $K_{n+1}(D^*(\widetilde{X})^\Gamma) \equiv \mathrm{S}^\Gamma_n(\widetilde{X})$ are defined as

$$\rho(\widetilde{D}) = [U^* \chi(\widetilde{D})_+] \text{ and } \rho(\widetilde{D}) = [\frac{1}{2}(1 + \chi(\widetilde{D})] \,.$$

(same definition as for $[D]$ but now no-need to work in the quotient because of the invertibility assumption).

- Notice that if n is odd then $\rho(\widetilde{D}) = [\Pi_{\geq}(\widetilde{D})]$
- we also get the universal rho classes $\rho_\Gamma(\widetilde{D}) := \widetilde{u}_* \rho(\widetilde{D})$ in $K_{n+1}(D^*_\Gamma) \equiv \mathrm{S}^\Gamma_*(E\Gamma)$
- Main example: if g is a positive scalar curvature metric and X is spin, then we have $\rho(g) := \rho(\widetilde{D}^{\mathrm{spin}}_g)$

Back to the Cheeger-Gromov rho invariant

- so we have defined the rho class in a strange K-theory group $\mathrm{S}^{\Gamma}_{*}(\widetilde{X}) := K_{*+1}(D^*(\widetilde{X})^{\Gamma})$!
- Let X be odd dimensional and let $\widetilde{D}$ be L^2-invertible
- Benemeur and Roy prove that there exists a homorphism of abelian groups $\tau_{\mathrm{CG}} : \mathrm{S}^{\Gamma}_{1}(\widetilde{X}) \to \mathbb{R}$ such that

$$\tau_{\mathrm{CG}}(\rho(\widetilde{D})) = \rho_{\mathrm{CG}}(\widetilde{D}) := \eta_{\Gamma}(\widetilde{D}) - \eta(D)$$

  the Cheeger-Gromov rho invariant
- All things considered this seems to be the right rho class !
- now we want to define higher rho numbers