K-Theoretic and homological invariants of Dirac operators. Lecture 3.

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Groupoids from a measurable, topological and geometric perspective

Outline of the third lecture

Summary of the first 2 lectures

K-homology

Index formulas

Higher rho invariants

Summary

- we have lifted our Dirac operator D to a Γ-equivariant operator $\widetilde{D}$ on $\widetilde{X}$, a Galois Γ-covering
- we have defined the compactly supported index class $\mathrm{Ind}_{\Gamma,c}(\widetilde{D}) \in K_0(\Psi^{-\infty}_{\Gamma,c}(\widetilde{X}, \widetilde{E}))$
- $\Psi^{-\infty}_{\Gamma,c}(\widetilde{X}, \widetilde{E})$ is the algebra of Γ-equivariant smoothing operators of Γ-compact support
- we have defined the C^*-index class $\mathrm{Ind}_{\Gamma}(\widetilde{D}) \in K_0(C^*(\widetilde{X}, \widetilde{E})^{\Gamma})$
- we have defined higher (compactly supported) indices $\langle \mathrm{Ind}_{\Gamma,c}(\widetilde{D}), \tau_{[\alpha]} \rangle$, with $[\alpha] \in H^*(\Gamma)$ and $\tau_{[\alpha]} \in HC^{\mathrm{even}}(\Psi^{-\infty}_{\Gamma,c}(\widetilde{X}, \widetilde{E}))$
- under additional assumptions on Γ we have defined higher C^*-indices

Summary (cont)

- for invertible operators we have defined a numeric secondary invariant, the Cheeger-Gromov rho invariant
- this is a secondary analogue of Atiyah's Γ-index $\mathrm{ind}_\Gamma(\widetilde{D})$
- we wanted to define a rho-class of an invertible $\widetilde{D}$ and then define higher rho numbers
- to this end we have introduced the Higson-Roe analytic surgery sequence
- let us not write the vector bundle $\widetilde{E}$
- $$\cdots \to K_{*+1}(C^*_r\Gamma) \to \mathrm{S}^\Gamma_*(\widetilde{X}) \to K_*(X) \xrightarrow{\delta} K_*(C^*_r\Gamma) \to \cdots$$
  with
- $\mathrm{S}^\Gamma_*(\widetilde{X}) := K_{*+1}(D^*(\widetilde{X})^\Gamma)$ the analytic structure group
- we have defined $[D] \in K_*(X)$ and $\mathrm{Ind}_{\mathrm{Roe}}(\widetilde{D}) \in K_{*+1}(C^*(\widetilde{X})^\Gamma)$
- $\mathrm{Ind}_{\mathrm{Roe}}(\widetilde{D}) = \mathrm{Ind}_\Gamma(\widetilde{D}) \in K_{*+1}(C^*(\widetilde{X})^\Gamma)$
- if $\widetilde{D}$ is invertible we have defined the rho class $\rho(\widetilde{D}) \in \mathrm{S}^\Gamma_*(\widetilde{X})$
- end of summary

Debts

before we proceed I want to pay a few debts

- debt 1: what is K-homology ?
- debt 2: Atiyah-Singer index formula
- debt 3: Connes-Moscovici higher index formula
- debt 4: why "surgery" ?

what is K-homology ?

- X compact Hausdorff space
- $K_*(X) := KK_0(C(X), \mathbb{C})$
- we define $KK_0(A, \mathbb{C})$ in general (A is a C^*-algebra)
- $KK_0(A, \mathbb{C})$ is defined as equivalence classes of **graded** Fredholm modules
- a graded Fredholm module is a triple (H, ϕ, F) with $H = H^+ \oplus H^-$ a $\mathbb{Z}_2$ graded Hilbert **space**, $\phi : A \to \mathcal{B}(H)$ a representation, $\phi = \phi^+ \oplus \phi^-$, and $F : H \to H$ an **odd** bounded operator $F = \begin{pmatrix} 0 & F^+ \\ F^- & 0 \end{pmatrix}$. with the property that

  $$(F^2 - 1)\phi(a)\,, \quad [F, \phi(a)]\,, \quad (F - F^*)\phi(a)$$

  are all **compact** operators
- we consider unitary classes of such triples
- we impose a equivalence relation through operator-homotopy
- this is $KK_0(A, \mathbb{C})$; for $KK_1(A, \mathbb{C})$ forget the grading !
- there is also an unbounded picture

Key example

- $X^{2\ell}$ is a smooth orientable compact m. without boundary
- g is a riemannian metric, D a Dirac type operator acting on the sections of a vector bundle $E = E^+ \oplus E^-$
- we consider $H := L^2(X, E)$, the Hilbert space of L^2-sections
- $L^2(X, E) = L^2(X, E^+) \oplus L^2(X, E^-)$
- D is $\mathbb{Z}_2$-graded **odd** operator with respect to $E^\pm$
- let $\phi : C(X) \to \mathcal{B}(L^2(X, E))$ be given by **multiplication**
- $\left(L^2(X, E), \phi, D/(1 + D^2)^{1/2}\right)$ defines an element $[D]$ in $KK_0(C(X), \mathbb{C})$ in the **bounded picture**
- $\left(L^2(X, E), \phi, D\right)$ defines the same element $[D]$ in $KK_0(C(X), \mathbb{C})$ in the **unbounded picture**.
- proof based on **classic elliptic theory**
- if X is odd dimensional, then E is ungraded and D defines $[D] \in KK_1(C(X), \mathbb{C})$
- these classes do not depend on the choice of metrics etc
- these classes are compatible with the Higson-Roe classes through Paschke duality

The Atiyah-Singer index theorem.

- one of the great theorems in modern Mathematics
- Atiyah-Singer index formula
  $\mathrm{ind}\, D^+ = \int_M AS(R^M, R^E) = \, < [AS(R^M, R^E)], [M] >$
- Right hand side is topological and sometimes even homotopical
- Geometric applications for Gauss-Bonnet, signature and Dolbeault:
- FIRST, you prove by the Hodge-de Rham-Dolbeault theorem that $\chi(M) = \mathrm{ind}(d + d^*)^+$; $\mathrm{sign}(M) = \mathrm{ind}\, D^{+,\mathrm{sign}}$;
  $\chi(M, \mathcal{O}) = \mathrm{ind}(\overline{\partial} + \overline{\partial}^*)^+$
- THEN apply Atiyah-Singer and get magically Chern-Gauss-Bonnet, Hirzebruch and Riemann-Roch:

$$\chi(M) = \int_M \mathrm{Pf}(M); \ \mathrm{sign}(M) = \int_M L(M); \ \chi(M, \mathcal{O}) = \int_M \mathrm{Td}(M)$$

The Atiyah-Singer index theorem (cont).

- Assume that M^{4k} is spin; then Atiyah-Singer says $\mathrm{ind} \not{D}^{+} = \int_M \widehat{A}(M)$
- recall that the metric g is of positive scalar curvature (psc) if $\mathrm{scal}_g(x) > 0 \ \forall x$
- if g is psc then $\not{D}$ is invertible because we know that: $\not{D}^2 = \nabla\nabla^* + \mathrm{scal}_g/4$ (Lichnerowicz formula)
- it follows that the topological term $\int_M \widehat{A}(M)$ must be zero
- $\Rightarrow$ obstruction to existence of positive scalar curvature metrics.
- (needless to say, the whole C^*-index class $\mathrm{Ind}_\Gamma(\widetilde{D}) \in K_{*+1}(C^*(\widetilde{M})^\Gamma)$ is an obstruction, but the point is that $\int_M \widehat{A}(M)$ is highly computable.)

The Connes-Moscovici higher index theorem

- let $[\alpha] \in H^*(\Gamma) \equiv H^*(B\Gamma)$
- it defines an element $\tau_{[\alpha]} \in HC^*(\Psi^{-\infty}(\widetilde{X}, \widetilde{E}))$
- consider the higher index $\langle \mathrm{Ind}_{\Gamma,c}(\widetilde{D}), \tau_{[\alpha]} \rangle$
- Connes-Moscovici: if $r : X \to B\Gamma$ is the classifying map for $\widetilde{X}$ then
$$\langle \mathrm{Ind}_{\Gamma,c}(\widetilde{D}), \tau_{[\alpha]} \rangle = \int_X AS(R^X, R^E) \wedge r^*[\alpha]$$
- if Γ is Gromov hyperbolic or of polynomial growth then this holds for the higher C^*-indices
- for the spin-Dirac operator this gives $\int_X \widehat{A}(X) \wedge r^*[\alpha]$
- for the signature operator we obtain $\int_X L(X) \wedge r^*[\alpha]$

The Connes-Moscovici higher index theorem (cont)

- the higher $\widehat{A}$-genera are the collection of numbers

$$\{\int_X \widehat{A}(X) \wedge r^*[c]\,,\ \ [c] \in H^*(B\Gamma, \mathbb{R})\}$$

- if Γ is Gromov hyperbolic or of polynomial growth than these are higher obstructions to the existence of a psc metric
- indeed $\mathrm{Ind}_\Gamma(\widetilde{D}) = 0$ and so $\langle \mathrm{Ind}_\Gamma(\widetilde{D}), \tau_{[\alpha]} \rangle = 0$
- Example: for the torus T^n we have $\pi_1(T^n) = \mathbb{Z}^n$ and the classifying space is still a torus, the dual torus $(T^n)^*$
- the higher index corresponding to the volume form of $(T^n)^*$ is non-zero
- $\Rightarrow$ the torus does not admit a metric of psc !
- notice that the numeric index of the torus is zero, and so does not help in proving this result

The Novikov conjecture

the higher signatures are the collection of numbers

$$\{ \int_X L(X) \wedge r^*[c] \,,\ \ [c] \in H^*(B\Gamma, \mathbb{R}) \}$$

- Novikov conjecture: all the higher signatures are homotopy invariants.
- if Γ is Gromov hyperbolic or of polynomial growth this is true
- indeed, under these additional assumptions on Γ these are higher C^*-indices and we know that $\mathrm{Ind}_\Gamma(\widetilde{D})$ is a homotopy invariant (Kasparov, Hilsum-Skandalis)

Why " surgery " in the Higson-Roe surgery sequence ?

Consider the surgery exact sequence in topology (Browder, Novikov, Sullivan, Wall):

$$L_{n+1}(\mathbb{Z}\Gamma) \dashrightarrow \mathcal{S}(X) \to \mathcal{N}(X) \to L_n(\mathbb{Z}\Gamma)$$

Theorem (Higson-Roe, here in a version proved by P-Schick)
X orientable, closed, of dimension n; $\pi_1(X) = \Gamma$. There exists maps $\mathrm{Ind}_\Gamma, \rho, \beta$ such that

$$\begin{tikzcd}
L_{n+1}(\mathbb{Z}\Gamma) \arrow[r, dashed] \arrow[d, "\mathrm{Ind}_\Gamma"] & \mathcal{S}(X) \arrow[r] \arrow[d, "\rho"] & \mathcal{N}(X) \arrow[r] \arrow[d, "\beta"] & L_n(\mathbb{Z}\Gamma) \arrow[d, "\mathrm{Ind}_\Gamma"] \\
K_{n+1}(C^*_r\Gamma) \arrow[r] & \mathrm{S}^\Gamma_n(\widetilde{X}) \arrow[r] & K_n(X) \arrow[r] & K_n(C^*_r\Gamma)
\end{tikzcd}$$

is commutative

Why " surgery " ? (continuation)

We consider Stolz' sequence for positive scalar curvature metrics.
Let Z be such that $\pi_1(Z) = \Gamma$ (for example: $Z = X$, or $Z = B\Gamma$):

$$\cdots \to \Omega^{\mathrm{spin}}_{n+1}(Z) \to R^{\mathrm{spin}}_{n+1}(Z) \to \mathrm{Pos}^{\mathrm{spin}}_{n}(Z) \to \Omega^{\mathrm{spin}}_{n}(Z) \to \cdots$$

Theorem (P-Schick 2014) *There exists group homomorphisms* $\mathrm{Ind}_\Gamma, \rho, \beta$ *such that*

$$\begin{array}{ccccccc}
\Omega^{\mathrm{spin}}_{n+1}(Z) & \longrightarrow & R^{\mathrm{spin}}_{n+1}(Z) & \longrightarrow & \mathrm{Pos}^{\mathrm{spin}}_{n}(Z) & \longrightarrow & \Omega^{\mathrm{spin}}_{n}(Z) \\
\downarrow \beta & & \downarrow \mathrm{Ind}_\Gamma & & \downarrow \rho & & \downarrow \beta \\
K_{n+1}(Z) & \longrightarrow & K_{n+1}(C^*_r\Gamma) & \longrightarrow & \mathrm{S}^\Gamma_n(\widetilde{Z}) & \longrightarrow & K_n(Z)
\end{array}$$

is commutative

The delocalized Atiyah-Patodi-Singer index theorem

- crucial in this all business is the **delocalized Atiyah-Patodi-Singer index theorem in K-theory**
- Let $\widetilde{W}$ be an oriented even dimensional manifold with free cocompact Γ-action and with boundary $\partial\widetilde{W} = \widetilde{M}$.
- $\widetilde{D}$ a Γ-equivariant Dirac operator on $\widetilde{W}$
- assume that the boundary operator $\widetilde{D}_\partial$ is L^2-invertible

Theorem

(delocalized APS index theorem in K-theory, P-Schick 2013) There exists an index class $\mathrm{Ind}_\Gamma(\widetilde{D}) \in K_*(C^*(W)^\Gamma)$ *and*

$$\iota_*(\mathrm{Ind}_\Gamma(\widetilde{D})) = j_*(\rho(\widetilde{D}_\partial)) \quad \textit{in} \quad K_0(D^*(\widetilde{W})^\Gamma).$$

Here $j\colon D^*(\partial\widetilde{W})^\Gamma \to D^*(\widetilde{W})^\Gamma$ *is induced by the inclusion* $\partial\widetilde{W} \hookrightarrow \widetilde{W}$ *and* $\iota\colon C^*(\widetilde{W})^\Gamma \to D^*(\widetilde{W})^\Gamma$ *is the natural inclusion.*

Back to higher rho invariants

- back to higher rho numbers
- we want to define higher rho numbers by pairing the rho class with suitable cyclic cohomology groups
- first of all, which cyclic cohomology groups ?!
- for the Index class $\mathrm{Ind}(\widetilde{D})$ we've Connes and Moscovici: if Γ is Gromov hyperbolic or of polynomial growth then $\langle\, ,\, \rangle : K_*(C^*(\widetilde{X})) \otimes H^*(\Gamma, \mathbb{C}) \to \mathbb{C};\ x \otimes [\alpha] \to \langle x, \tau_{[\alpha]} \rangle$
- consider $HC^*(\mathbb{C}\Gamma)$
- Burghelea showed that $HC^*(\mathbb{C}\Gamma)$ decomposes as the direct product of $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$
- here $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ is defined requiring $\tau(g_0, g_1, \ldots, g_k) = 0$ if $g_0 \cdots g_k \notin \langle x \rangle$.
- Moreover one proves that $HC^*(\mathbb{C}\Gamma, \langle e \rangle) = H^*(\Gamma, \mathbb{C})$
- so Connes and Moscovici is really a pairing $\langle\, ,\, \rangle : K_*(C^*(\widetilde{X})) \otimes HC^*(\mathbb{C}\Gamma, \langle e \rangle) \to \mathbb{C}$

Higher rho invariants (cont)

- the index class is "localized at the diagonal" (because it is defined in terms of parametrices that can be localized near the diagonal)
- the rho class $\rho(\widetilde{D})$ in $\mathrm{S}^{\Gamma}_{*}(\widetilde{X})$ is not localized (recall, for example, that if X is odd dimensional, then $\rho(\widetilde{D}) = [\Pi_{>}(\widetilde{D})]$)
- we are led to pair $\mathrm{S}^{\Gamma}_{*}(\widetilde{M})$ with $HC^{*}(\mathbb{C}\Gamma, \langle x \rangle)$, the delocalized cyclic cohomology of $\mathbb{C}\Gamma$
- we shall see later why this guess is the good one
- one sees immediately that the algebra $D^{*}(\widetilde{X})$ is "too big"
- here is where groupoids enter the scene