K-Theoretic and homological invariants of Dirac operators. Lecture 4.

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Groupoids from a measurable, topological and geometric perspective

Outline of the fourth lecture

Summary of the first 3 lectures

Higher rho invariants

A glimpse to some geometric applications

Summary

- we have defined the compactly supported index class $\mathrm{Ind}_{\Gamma,c}(\widetilde{D}) \in K_0(\Psi^{-\infty}_{\Gamma,c}(\widetilde{X},\widetilde{E}))$
- we have defined the C^*-index class $\mathrm{Ind}_\Gamma(\widetilde{D}) \in K_0(C^*(\widetilde{X},\widetilde{E})^\Gamma)$
- we have defined higher (compactly supported) indices $\langle \mathrm{Ind}_{\Gamma,c}(\widetilde{D}), \tau_{[\alpha]}\rangle$, with $[\alpha] \in H^*(\Gamma)$ and $\tau_{[\alpha]} \in HC^{\mathrm{even}}(\Psi^{-\infty}_{\Gamma,c}(\widetilde{X},\widetilde{E}))$
- under additional assumptions on Γ we have defined higher C^*-indices

Summary (cont)

- we introduced the Higson-Roe analytic surgery sequence (let us not write the vector bundle $\widetilde{E}$)
- $$\cdots \to K_{*+1}(C^*_r\Gamma) \to \mathrm{S}^{\Gamma}_*(\widetilde{X}) \to K_*(X) \xrightarrow{\delta} K_*(C^*_r\Gamma) \to \cdots$$ with
- $\mathrm{S}^{\Gamma}_*(\widetilde{X}) := K_{*+1}(D^*(\widetilde{X})^{\Gamma})$ the analytic structure group
- we have defined $[D] \in K_*(X)$ and $\mathrm{Ind}_{\mathrm{Roe}}(\widetilde{D}) \in K_{*+1}(C^*(\widetilde{X})^{\Gamma})$
- $\mathrm{Ind}_{\mathrm{Roe}}(\widetilde{D}) = \mathrm{Ind}_{\Gamma}(\widetilde{D}) \in K_{*+1}(C^*(\widetilde{X})^{\Gamma})$
- if $\widetilde{D}$ is invertible we have defined the rho class $\rho(\widetilde{D}) \in \mathrm{S}^{\Gamma}_*(\widetilde{X})$
- now we want to extract higher rho numbers out of the rho class

Higher rho invariants

- recall: Burghelea showed that $HC^*(\mathbb{C}\Gamma)$ decomposes as the direct product of $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$
- here $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ is defined requiring $\tau(g_0, g_1, \dots, g_k) = 0$ if $g_0 \cdots g_k \notin \langle x \rangle$.
- Moreover $HC^*(\mathbb{C}\Gamma, \langle e \rangle) = H^*(\Gamma, \mathbb{C})$
- we would like to pair $\mathrm{S}^\Gamma_*(\widetilde{M})$ with $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ a costituent of the delocalized cyclic cohomology of $\mathbb{C}\Gamma$
- here is where we use groupoids

Higson-Roe via groupoids

Zenobi in his Ph.D. thesis proposed a different approach to

$$[D] \in K_*(X), \;\; \mathrm{Ind}_\Gamma(\widetilde{D}) \in K_*(C^*(\widetilde{X})^\Gamma), \;\; \rho(\widetilde{D}) \in K_{*+1}(D^*(\widetilde{D})^\Gamma)$$

and to

$$\cdots \to K_{*+1}(C^*_r\Gamma) \to \mathrm{S}^\Gamma_*(\widetilde{X}) \to K_*(X) \xrightarrow{\delta} K_*(C^*_r\Gamma) \to \cdots$$

To minimize the use of tilde let me denote $\widetilde{X}$ by X_Γ and $\widetilde{D}$ by D_Γ

- let us make a few preliminary remarks
- **1.** $K^*(T^*X)$ is isomorphic to $K_*(X)$ through an explicit isomorphism $K^*(T^*X) \xrightarrow{q} K_*(X)$
- this isomorphism q is "Poincaré duality in K-Theory".
- for example: elements in $K^0(T^*X) = K^0(D^*X, S^*X)$ are given by elliptic symbols; the map q is the quantization map, it is the map $[\sigma] \to [P_\sigma]$ (it associates to a symbol σ the analytic K-homology class of the operator P_σ defined by σ).
- **2.** Any vector bundle $E \xrightarrow{p} X$ is a groupoid with units equal to X and with $r = s = p$; hence TX has a groupoid structure.

The adiabatic groupoid

The first object of interest is the following Lie groupoid with units X:

$$G(X) = X_\Gamma \times_\Gamma X_\Gamma \rightrightarrows X$$

where

$$X_\Gamma \times_\Gamma X_\Gamma = X_\Gamma \times X_\Gamma / \Gamma$$

$r[x, y] = \pi(x)$ and $s[x, y] = \pi(y)$ with $\pi : X_\Gamma \to X$ the Galois covering projection.

Every Lie groupoid $G \rightrightarrows X$ has a Lie algebroid $\mathfrak{A}$, which is in particular a vector bundle over X endowed with an *anchor map* to TX: the Lie algebroid of $G(X)$ is TX and the ancor map is the identity.

Alain Connes introduced the adiabatic deformation of $G(X)$:

$$G(X)_{ad} := TX \times \{0\} \sqcup X_\Gamma \times_\Gamma X_\Gamma \times (0, 1] \rightrightarrows X \times [0, 1];$$

One can equip this set with a topology and a smooth structure. Thus we have obtained a new Lie groupoid, the adiabatic deformation of $G(X)$.

The long exact sequence of the adiabatic groupoid

- We know that we can associate a C^*-algebra to a Lie groupoid
- The C^* algebra of our groupoid $X_\Gamma \times_\Gamma X_\Gamma \rightrightarrows X$ is precisely the Roe algebra $C^*(X_\Gamma)^\Gamma$.
- we also have the C^*-algebra $C^*(TX)$ (viewing $TX \to X$ as a groupoid); there is an isomorphism of C^*-algebras $C^*(TX) \simeq C_0(T^*X)$

If we denote the restriction of $G(X)_{ad}$ to $X \times [0,1)$ by $G(X)^0_{ad}$, then using the evaluation at 0 we have

$$0 \longrightarrow C^*(X_\Gamma \times_\Gamma X_\Gamma \times (0,1)) \longrightarrow C^*(G(X)^0_{ad}) \xrightarrow{ev_0} C^*(TX) \longrightarrow 0$$

We then have a long exact sequence in K-Theory

$$\cdots \to K_*(C^*(TX)) \xrightarrow{\delta^{ad}} K_{*+1}(C^*(X_\Gamma \times_\Gamma X_\Gamma \times (0,1))) \to \cdots$$

Fundamental class and index class in the adiabatic context

- Starting with the principal symbol of D we can define a class $[\sigma_{\mathrm{pr}}(D)]$ in $K^*(T^*X) = K_*(C_0(T^*X)) = K_*(C^*(TX))$
- so $[\sigma_{\mathrm{pr}}(D)] \in K_*(C^*(TX))$ is the same as $[D] \in K_*(X)$ via Poincaré duality
- from the long exact sequence in K-Theory
  $\cdots \to K_*(C^*(TX)) \xrightarrow{\delta^{ad}} K_{*+1}(C^*(X_\Gamma \times_\Gamma X_\Gamma \times (0,1))) \to \cdots$
  we can define the adiabatic index class as usual as
  $\delta^{ad}[\sigma_{\mathrm{pr}}(D)] \in K_{*+1}(C^*(X_\Gamma \times_\Gamma X_\Gamma \times (0,1)))$
- prefer to define
  $\mathrm{Ind}^{ad}(D^\Gamma) := \beta \circ \delta^{ad}[\sigma_{\mathrm{pr}}(D)] \in K_*(C^*(X_\Gamma \times_\Gamma X_\Gamma)) = K_*(C^*_r\Gamma)$
  with $\beta =$ the Bott iso.
- one can prove that $\mathrm{Ind}^{ad}(D^\Gamma) = \mathrm{Ind}_\Gamma(D^\Gamma)$, our old friend.
- Summarizing: $[D] \leftrightarrow [\sigma_{\mathrm{pr}}(D)]$ and $\mathrm{Ind}_\Gamma(D^\Gamma) \leftrightarrow \mathrm{Ind}^{ad}(D^\Gamma)$

The rho class

- we make a few preliminary remarks
- given a Lie groupoid one can define $\Psi_c^k(G)$, $k \in \mathbb{Z}$
- there is a symbol map with values in the cosphere bundle of the Lie algebroid $\mathfrak{A}$
- there is also a quantization map associating to a symbol on $\mathfrak{A}^*$ an element in $\Psi_c^*(G)$
- ellipticity has the usual meaning
- an elliptic operator P defines a class in $KK_*(\mathbb{C}, C^*(G)) = K_*(C^*(G))$
- if U is a closed G-invariant set in $X := G^{(0)}$ then we can consider $G|_U$; if P restricted to $G|_U$ is invertible then we obtain a class in $KK_*(\mathbb{C}, C^*(G|_{X \setminus U})) = K_*(C^*(G|_{X \setminus U})$

The rho class (cont)

Now, if D^Γ is invertible we shall define an adiabatic rho class $\rho^{ad}(D^\Gamma) \in K_*(C^*(G(X)^0_{ad}))$.

$$K_*(C^*(G(X)^0_{ad})) \xrightarrow{\mathrm{ev}_0} K_*(C^*(TX)) \xrightarrow{\mathsf{Ind}^{\mathrm{ad}}} K_*(C^*(X_\Gamma \times_\Gamma X_\Gamma))$$

$$\rho^{ad}(D) \dashleftarrow [\sigma_{\mathrm{pr}}(D)] \longrightarrow 0$$

- How do we achieve this ?!
- (i) we consider $\sigma(D) \otimes \mathrm{Id}_{[0,1]}$
  (ii) we consider the operator on $G_{ad} \rightrightarrows X \times [0,1]$ obtained by quantization from this symbol
  (iii) this operator is invertible when restricted to $X \times \{1\}$ (it's our invertibility assumption)
  (iv) it defines a class in the K-theory of the C^*-algebra of the "complementary part", that is in $K_*(C^*(G(X)^0_{ad}))$
- this last class is the adiabatic rho class $\rho^{ad}(D^\Gamma)$!!

From Zenobi to Higson-Roe

Prior work of several people and work of Zenobi show that
(i) there are maps from the adiabatic long exact sequence in K-theory to the Higson-Roe surgery sequence
(ii) these maps are isomophisms and make the diagram commute
(iii) they transform

$$[\sigma(D)], \quad \mathrm{Ind}^{ad}(D^\Gamma), \quad \rho^{ad}(D^\Gamma)$$

into

$$[D], \quad \mathrm{Ind}_\Gamma(D^\Gamma), \quad \rho(D^\Gamma)\,.$$

Very important: this works for any groupoid $G \rightrightarrows M$ with algebroid $\mathfrak{A}$, with $G_{ad} := \mathfrak{A} \times \{0\} \sqcup G \times (0,1] \rightrightarrows M \times [0,1]$

Higher rho invariants: short story

- Using Zenobi's description of the structure group $\mathrm{S}^{\Gamma}_{*}(\widetilde{X})$ we prove among other things the following result

Theorem
Let Γ be Gromov hyperbolic. Then there is a well defined pairing

$$\langle\ ,\ \rangle : \mathrm{S}^{\Gamma}_{*}(\widetilde{X}) \times HC^{*}(\mathbb{C}\Gamma, \langle x \rangle) \to \mathbb{C}$$

- this is substantially more difficult than Connes-Moscovici
- applying it to $\rho(\widetilde{D})$ we can define higher rho numbers

$$\rho^{\tau}(\widetilde{D}) := \langle \rho(\widetilde{D}), \tau \rangle\,, \quad \tau \in HC^{*}(\mathbb{C}\Gamma, \langle x \rangle).$$

For example $\rho^{\tau}(g)$, the higher rho number of a PSC metric g (through the spin-Dirac operator associated to g).

- There are explicit formulae.
- For example: if M is odd dimensional and τ is the delocalized trace $\tau_{\langle x\rangle} \in HC^0(\mathbb{C}\Gamma, \langle x\rangle)$:

$$\tau_{\langle x\rangle}(\sum \alpha_\gamma \gamma) := \sum_{\gamma \in \langle x\rangle} \alpha_\gamma$$

  then the corresponding rho number is Lott's delocalized eta invariant $\eta_{\langle x\rangle}(\widetilde{D})$:

$$\rho^{\tau_{\langle x\rangle}}(\widetilde{D}) = \eta_{\langle x\rangle}(\widetilde{D})$$

Lott's delocalized eta invariant

Fix a non-trivial conjugacy class $\langle x \rangle$ of Γ.

$$\mathrm{Tr}_{\langle x \rangle}(\widetilde{D} \exp(-(t\widetilde{D})^2)) := \sum_{h \in \langle x \rangle} \int_{\mathcal{F}} \mathrm{tr}_p \, \widetilde{K}_t(p, hp).$$

This is called the **delocalized trace**. If the conjugacy class $\langle x \rangle$ if finite then the **delocalized eta invariant** is defined as:

$$\eta_{\langle x \rangle}(\widetilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^{\infty} \mathrm{Tr}_{\langle x \rangle}(\widetilde{D} \exp(-(t\widetilde{D})^2) \, dt.$$

Problems at $t = \infty$ if $|\langle x \rangle| = \infty$.

Pushnigg: OK if Γ is Gromov hyperbolic.

Strategy (very brief)

Step 1: we map the entire adiabatic sequence to a sequence in non-commutative de Rham homology, with commutative squares. No assumptions on Γ !

Theorem. (P-Schick-Zenobi) *For any subalgebra $\mathcal{A}\Gamma$ of $C_r^*\Gamma$ dense and holomorphically closed there exist Chern characters*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & K_{*-1}(C^*_{red}\Gamma) & \longrightarrow & \mathrm{S}^{\Gamma}_{*}(\widetilde{X}) & \longrightarrow & K_*(X) & \longrightarrow \cdots \\ & \downarrow \mathrm{Ch}_{\Gamma} & & \downarrow \mathrm{Ch}^{del}_{\Gamma} & & \downarrow \mathrm{Ch}^{e}_{\Gamma} & \\ \cdots \xrightarrow{j_*} & H_{[*-1]}(\mathcal{A}\Gamma) & \longrightarrow & H^{del}_{[*-1]}(\mathcal{A}\Gamma) & \xrightarrow{\delta} & H^{e}_{[*]}(\mathcal{A}\Gamma) & \xrightarrow{j_*} \cdots \end{array}$$

making the diagram commute.

Proof employs Zenobi's description of the surgery sequence.

Step 2: we pair the de Rham sequence of Step 1 with $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ if Γ is Gromov hyperbolic or of polynomial growth.

Non-commutative de Rham homology

• Recall that if A is a unital algebra then $H_*(A)$ is the homology of $\Omega_*(A)_{ab} := \Omega_*(A)/[\Omega_*(A), \Omega_*(A)]$ with $\Omega_*(A)$ the universal differential graded algebra (universal DGA).
• we write an element in $\Omega_*(A)$ as $\sum a_0 da_1 \dots da_k$
• If A is a Fréchet algebra we take projective tensor products and the closure of the graded commutators (denoted $\widehat{\Omega}(A)_{ab}$)
• In particular we have $H_*(\mathbb{C}\Gamma)$ and $H_*(\mathcal{A}\Gamma)$ with $\mathcal{A}\Gamma$ a subalgebra of $C_r^*\Gamma$ dense and holomorphically closed.
• A basis for $\Omega_*(\mathbb{C}\Gamma)$ is given by $\{g_0 dg_1 \cdots dg_k\}$
• the inclusion $j : \mathbb{C}\Gamma \hookrightarrow \mathcal{A}\Gamma$ induces a morphism of DGA $j : \Omega_*(\mathbb{C}\Gamma) \to \widehat{\Omega}_*(\mathcal{A}\Gamma)$
• $\Omega^e(\mathbb{C}\Gamma)$ is the sub-DGA generated by $\{g_0 dg_1 \cdots dg_k\}$ such that $g_0 g_1 \cdots g_k = e$
• we consider the closure of $j(\Omega^e(\mathbb{C}\Gamma)_{ab})$ in $\widehat{\Omega}_*(\mathcal{A}\Gamma)_{ab}$; get $H_*^e(\mathcal{A}\Gamma)$. Using the quotient complex we also get $H_*^{del}(\mathcal{A}\Gamma)$.

Higher rho numbers: Step 2

- Let M be odd dimensional. Assume that $\widetilde{D}$ is L^2-invertible.
- We have the rho class $\rho(\widetilde{D}) \in \mathrm{S}_1^{\Gamma}(\widetilde{M})$ and we've defined $\mathrm{Ch}_{\Gamma}^{del}(\rho(\widetilde{D})) \in H_{\mathrm{ev}}^{del}(\mathcal{A}\Gamma) \subset H_{\mathrm{ev}}(\mathcal{A}\Gamma)$
- It is a general fact that $H_*(\mathcal{A}\Gamma)$ embeds in $HC_*(\mathcal{A}\Gamma)$
- We could pair $\mathrm{Ch}_{\Gamma}^{del}(\rho(\widetilde{D})) \in H_{\mathrm{ev}}^{del}(\mathcal{A}\Gamma)$ with $HC^*(\mathcal{A}\Gamma)$ but this group is very difficult to compute
- Instead, we would like to pair this class with $HC^*(\mathbb{C}\Gamma, \langle x \rangle)$, $x \neq e$.
- This is an **extension** problem exactly as for Connes and Moscovici
- Given $[\tau] \in HC^*(\mathbb{C}\Gamma, \langle x \rangle)$ we would like to extend it to a class in $HC^*(\mathcal{A}\Gamma)$ and then use $H_*(\mathcal{A}\Gamma) \times HC^*(\mathcal{A}\Gamma) \to \mathbb{C}$
- this would give a sense to $\langle \mathrm{Ch}_{\Gamma}^{del}(\rho(\widetilde{D})), [\tau] \rangle$

Higher rho numbers: Gromov hyperbolic groups

We assume Γ Gromov hyperbolic.

We want to extend **delocalized** cocycles.

Note that this is a difficult problem already for the delocalized trace associated to $\langle x \rangle$, $\tau_{\langle x \rangle}(\sum_\gamma a_\gamma \gamma) = \sum_{g \in \langle x \rangle} a_g$

Theorem

(Puschnigg, 2010) Let Γ be Gromov hyperbolic. There exists a smooth subalgebra $\mathcal{A}\Gamma \subset C_r^*\Gamma$ s. t. $\tau_{\langle x \rangle}$ extends from $\mathbb{C}\Gamma$ to $\mathcal{A}\Gamma$.

Theorem B. (P-Schick-Zenobi, 2019) Let Γ be Gromov hyperbolic.

(1) $\forall x \in \Gamma$ there is an isomorphism

$$HC^*(\mathbb{C}\Gamma, \langle x \rangle) = HC^*_{\mathrm{pol}}(\mathbb{C}\Gamma, \langle x \rangle)$$

(2) The cyclic cochains of polynomial growth extends to the Puschnigg's algebra $\mathcal{A}\Gamma$ inducing an injection $HC^*(\mathbb{C}\Gamma, \langle x \rangle) \to HC^*(\mathcal{A}\Gamma)$ as a direct summand.

For (1) we build on results of Dan Burghelea and Ralph Meyer.

For (2) we use heavily the work of Michael Puschnigg.

Higher rho numbers: summary

- Summarizing: for a hyperbolic group we have finally established the existence of a pairing

$$\langle\ ,\ \rangle : \mathrm{S}^{\Gamma}_{*}(\widetilde{M}) \times HC^{*}(\mathbb{C}\Gamma, \langle x \rangle) \to \mathbb{C}$$

  given by $\langle x, \tau \rangle := \langle \mathrm{Ch}^{del}_{\Gamma}(x), \tau \rangle$
- taking $x = \rho(\widetilde{D})$ we can define higher rho numbers

$$\rho^{\tau}(\widetilde{D}) := \langle \mathrm{Ch}^{del}_{\Gamma}(\rho(\widetilde{D})), \tau \rangle\,, \quad \tau \in HC^{*}(\mathbb{C}\Gamma, \langle x \rangle).$$

  For example $\rho^{\tau}(g)$, the higher rho number of the spin-Dirac operator of a PSC metric g.
- all results hold also for Γ of polynomial growth
- a different approach to the definition of the higher rho numbers has been developed by Xiaoman Chen, Jinmin Wang, Zhizhang Xie, and Guoliang Yu.

Rho numbers and metrics of positive scalar curvature

- $\mathcal{R}^+(X) :=$ space of metrics of positive scalar curvature (psc)
- X spin
- We want to use $\rho^\tau(g)$, $\tau \in HC^*(\mathbb{C}\Gamma, \langle x \rangle)$, in order to study $\mathcal{P}^+(M)$, the space of concordance classes of metrics of psc
- g_0 is concordant to g_1 if there exists g of psc on $[0,1] \times X$ restricting to g_0 and g_1 on the boundary
- Important: $\rho^\tau(g)$ is well defined on concordance classes; this employs the APS delocalized index theorem in K-Theory
- In fact $\rho^\tau(g)$ is well defined in $\mathrm{Pos}_*^{rmspin}(X)$
- there is an action of $\mathrm{Diffeo}(X)$ on $\mathcal{P}^+(X)$
- Stolz: once we fix a base-metric $[g_0]$ there is a group structure on $\mathcal{P}^+(X)$
- we are interested in the *coinvariants* $\mathcal{P}^+(X)_U$ associated to $U \leq \mathrm{Diffeo}(X)$, a finite index subgroup
- E.g. : $U =$ spin preserving diffeomorphisms

- recall that if a group U acts on the left on an abelian group A, then $A_U := A/ < \{a - u \cdot a, a \in A, u \in U\} >$
- $\mathcal{P}^+(X)_U$ should be thought of as a **moduli space**
- we want to give a lower bound for the rank of $\mathcal{P}^+(X)_U$
- we have many results !! the next one is just an example
- Consider $F\Gamma := \{f : \Gamma_{fin} \to \mathbb{C} \mid |\text{supp}(f)| < \infty\}$ with Γ_{fin} denoting the elements of finite order
- Consider $F^p\Gamma = \{f \in F\Gamma \mid f(\gamma) = (-1)^p f(\gamma^{-1})\}$ for $p = 0, 1$
- Consider $F^p_{del}\Gamma := \{f \in F^p\Gamma \mid \sum f(g) = 0\}$

Theorem

(P-Schick-Zenobi 2021) Assume that Γ *is Gromov hyperbolic. Assume that* $\mathrm{Out}(\Gamma)$ *is finite (this is a weak assumption). Then there exists a finite index subgroup* $U \leq \mathrm{Diffeo}(M)$ *such that*

$$\mathrm{rank}(\mathcal{P}^+(M)_U) \geq \sum_{k>0, p\in\{0,1\}} \mathrm{rank}(H^{n+1-4k-2p}(\Gamma; F^p_{del}\Gamma)).$$

The cohomology groups $H^{n+1-4k-2p}(\Gamma; F^p_{del}\Gamma))$ can be explicitly computed in terms of centralizers of finite order elements.

THANK YOU!