Measure equivalence of Baumslag–Solitar groups

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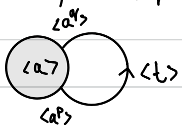
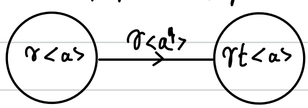
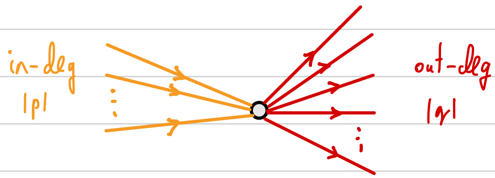
Baumslag-Solitar groups.

For nonzero $p,q \in \mathbb{Z}$, the associated Baumslag-Solitar group is $BS(p,q) := \langle a, t \mid t a^p t^{-1} = a^q \rangle$.
Thus, $BS(p,q)$ is an HNN-extension, with associated Bass-Serre tree $\mathcal{T}_{|p|,|q|} := (V_{|p|,|q|}, E_{|p|,|q|})$,

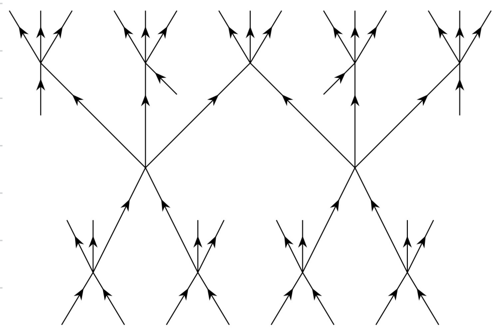


where $V_{|p|,|q|} = BS(p,q)/\langle a \rangle$
$E_{|p|,|q|} = BS(p,q)/\langle a^q \rangle$





$BS(p,q) \curvearrowright \mathcal{T}_{|p|,|q|}$ so $BS(p,q) \leq \mathrm{Aut}(p,q) := \mathrm{Aut}(\mathcal{T}_{|p|,|q|})$.
We say that $BS(p,q)$ is "unimodular" if $\mathrm{Aut}(p,q)$ is unimodular.



$\mathcal{T}_{2,3}$

The modular homomorphism $\Delta : \mathrm{Aut}(p,q) \to \mathbb{R}_{>0}$ is given by $\begin{cases} K \mapsto 1 \\ t \mapsto |p|/|q| \end{cases}$, where $K := \mathrm{St}(1\langle a \rangle)$.

Thus, $BS(p,q)$ is "unimodular" $\iff |p| = |q|$.

Partition into natural families.

(F1) Amenable: $|p| = 1$ or $|q| = 1$. These are solvable, hence amenable.
(F2) Nonamenable "unimodular": $|p| = |q| \neq 1$. These are virtually $\cong \mathbb{F} \times \mathbb{Z}$, where $\mathbb{F}$ is a f.g. nonabelian free gp.
(F3) Nonamenable "nonunimodular": $|p|, |q| > 1$ and $|p| \neq |q|$.

Measure equivalence (ME).

Measure equivalence is a measure-theoretic analogue of quasi-isometry expressed in terms of topological couplings.

Def (Gromov 1993). Ctbl groups Γ and Δ are *measure equivalent (ME)* if they admit a *measure coupling*: a standard σ-finite measure space (Ω, μ) equipped with commuting measure-preserving actions $\Gamma \curvearrowright \Omega \curvearrowleft \Delta$ which are free and admit fundamental domains of finite measure, i.e. $\exists$ Borel sets $A, B \subseteq \Omega$ s.t. $\bigsqcup_{\gamma \in \Gamma} \gamma \cdot A = \Omega = \bigsqcup_{\delta \in \Delta} B \cdot \delta$.

One can "compose" these couplings to show transitivity, hence ME is indeed an eq. rel.

Examples.

(a) If $\Gamma < \Delta$ has finite index, then Γ ME Δ. Hence, commensurability $\Rightarrow$ ME.

Proof. Take $\Omega := \Delta$ with $\Gamma \curvearrowright \Omega \curvearrowleft \Delta$ by translations. Then $|\Omega / \Gamma| = [\Delta : \Gamma] < \infty$. $\square$

(b) If Γ, Δ are cocompact lattices in the same lcsc group G, then Γ ME Δ. In fact,

$$\Gamma \text{ ME } G \text{ ME } \Delta.$$

Quasi-isometry (q.i.) and measure equivalence (ME) classifications.

Because ME is a sibling of q.i., let's compare them on Baumslag-Solitar groups.

Note that $BS(q,p) \cong BS(p,q) \cong BS(-p,-q)$ and by a result of Casals-Ruiz-Kazachkov-Zakharov, $BS(p,q)$ and $BS(p,-q)$ are commensurable. Thus, these are all q.i. and ME, so the q.i. and ME classification only needs to be explained for $1 \leq p \leq q$.

Quasi-isometry.

(F1) Farb-Mosher 1998: q.i. = commensurability: $BS(1,n^k) \sim BS(1,n^\ell)$, $n \geq 1$.

$\nparallel$ q.i. preserves amenability

(F2) one class

$\nparallel$ Whyte 2001

(F3) Whyte 2001: one class

Measure equivalence (= orbit equivalence, Kida 2014)

(F1) Ornstein-Weiss 1987: one class

$\nparallel$ ME preserves amenability

(F2) one class

$\nparallel$ Kida 2014

(F3) Gaboriau-Poulin-Ø.-Tucker-Drob-Wrobel 2025+: ?

Evidence for/against ME rigidity in (F3).

Kida 2014. For (F3), if $BS(p_0,q_0)$ ME $BS(p_1,q_1)$, then so are their modular kernels.

Hope? Maybe the modular kernels aren't ME, hence rigidity in (F3).

Kida 2024. Nope: the modular kernels are all ME to $\mathbb{F}_\infty \times \mathbb{Z}$.

Forester 2024. For $k \geq 2$, $BS(k,k^2)$ ME $BS(k^2,k^4)$ because they are virtually contained as cocompact lattices in the same lcsc group.

Hope? This is the first ME example that is not just commensurability. Maybe Forester's methods can be pushed further for other groups in (F3)?

Carette - Cornulier - Le Boudec - Tessera 2025+. Nope: if $\langle p_0/q_0 \rangle$ and $\langle p_1/q_1 \rangle$ are not commensurable as subgroups of $\mathbb{R}_{>0}$, then $BS(p_0,q_0)$ and $BS(p_1,q_1)$ are not commable; in particular, they are not virtually contained as cocompact lattices in the same lcsc group.

Kida 2014 + Houdayer - Raum 2015. For (F3), if a "very ergodic" measure coupling (Ω,μ) exists for $BS(p_0,q_0)$ and $BS(p_1,q_1)$, then $BS(p_0,q_0) \cong BS(p_1,q_1)$. "Very ergodic" means the eliptic subgps ($\langle a \rangle$ and its conjugates) of $BS(p_i,q_i)$ act ergodically on $\Omega / BS(p_{1-i},q_{1-i})$ for all $i \in \{0,1\}$.

Hope? Maybe one can boost any measure coupling into a special one, hence rigidity in (F3).

Final nope:

Gaboriau - Poulin - Ō. - Tucker-Drob - Wrobel 2025+. (F3) is one ME class, i.e. all groups $BS(p,q)$ with $|p|,|q|>1$ and $|p| \neq |q|$ are ME to each other.

The rest of the time will be spent discussing the scheme of the proof and its ingredients.

Scheme of Proof.

Let $1 < p_i < q_i$, $i = 0,1$.

$\boxed{BS(p_0,q_0) \text{ ME } BS(p_1,q_1)}$ Probability measure preserving (pmp) context.

$\Uparrow$

$\boxed{Aut(p_0,q_0) \text{ ME } Aut(p_1,q_1)}$ Because $Aut(p_i,q_i)$ are nonunimodular, ME for them is only a measure class preserving (mcp) notion.

$\Uparrow$

$\boxed{\exists \text{ free pmp actions } Aut(p_i,q_i) \curvearrowright (Y_i,\nu_i) \text{ with orbit equivalent cross-sections.}}$ Cross-section equivalence relations of pmp actions of nonunimodular groups are mcp equivalence relations, typically type III.

$\Uparrow$

Solve a problem about mcp measured treeings.

Here we exploit the recently developed theory for mcp graphs, in particular, the existence of ergodic hyperfinite subgraphs (T. 2022), as well as a new way of using the associated Krieger flow.

Proof steps and ingredients.

(1) $BS(p_0,q_0)$ ME $BS(p_1,q_1)$.

$BS(p,q) \hookrightarrow \mathrm{Aut}(p,q) \ltimes \mathbb{R}$ by $\begin{cases} a \mapsto (a,1) \\ t \mapsto (t,0) \end{cases}$ is a cocompact lattice.

(2) $\mathrm{Aut}(p_0,q_0) \ltimes \mathbb{R}$ ME $\mathrm{Aut}(p_1,q_1) \ltimes \mathbb{R}$, where $\mathrm{Aut}(p_i,q_i) \curvearrowright \mathbb{R}$ by dilations via Δ.

By Koivisto-Kyed-Raum, ME of $\mathrm{Aut}(p_i,q_i)$ is equivalent to the existence of free pmp actions $\mathrm{Aut}(p_i,q_i) \curvearrowright (Y_i,\nu_i)$ with isomorphic action groupoids. By Hahn's theory of Haar measures for measured groupoids, the modular functions must be conjugate...

(3) $\mathrm{Aut}(p_0,q_0)$ ME $\mathrm{Aut}(p_1,q_1)$.

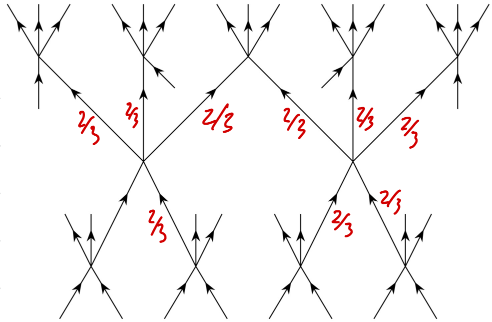
Koivisto-Kyed-Raum 2021

(4) $\exists$ free pmp $\mathrm{Aut}(p_i,q_i) \curvearrowright (Y_i,\nu_i)$ with mcp orbit equivalent cross-sections.

Let $K \leq \overbrace{\mathrm{Aut}(p,q)}^{G}$ be the stabilizer of a vertex in $T_{p,q}$ (it's compact open). Then a Borel transversal X for $K \curvearrowright Y$ is a cross-section, identified with Y/K. Build a $T_{p,q}$-treeing via the identification $[x]_{R_G} \simeq G/K \simeq V(T_{p,q})$. Converse also holds.

(5) $\exists$ mcp countable Borel equivalence relations R_i on (X_i,μ_i) which are mcp isomorphic and admit μ_i-homogeneous T_{p_i,q_i}-treeings.

A $T_{p,q}$-treeing is a directed Borel graph $\mathcal{G}$ on a standard probability space (X,μ) whose components are isomorphic to the tree $T_{p,q}$.
It is said to be μ-homogeneous if the Radon-Nikodym cocycle with respect to μ is equal to p/q on each edge $(x,y) \in T_{p,q}$.



Main theorem. Let R be an ergodic mcp ctbl Borel equivalence relation on (X,μ_0) which admits a μ_0-homogeneous T_{p_0,q_0}-treeing. If R is type III_0 and its Krieger flow admits a p_1/q_1-equispaced cross-section, then R also admits a μ_1-homogeneous T_{p_1,q_1}-treeing for some $\mu_1 \sim \mu_0$.

Apply this theorem to the cross-section (as above) of $\mathrm{Aut}(p_0,q_0) \curvearrowright (Z_0,\kappa_0) \times [1, p_1/q_1)$, where
- $\mathrm{Aut}(p_0,q_0) \curvearrowright (Z_0,\kappa_0)$ is any mixing free pmp action, e.g. Poisson point process on $\mathrm{Aut}(p_0,q_0)$
- $\mathrm{Aut}(p_0,q_0) \curvearrowright [1, p_1/q_1)$ via the modular homomorphism $\Delta_0 : \mathrm{Aut}(p_0,q_0) \to \mathbb{R}_{>0}$

Ingredients of the proof of Main Theorem.

Using the fact that in type III_0, the Krieger flow is aperiodic, we obtain:

Theorem 1. Every type III_0 ergodic ctbl Borel eq. rel. R is hyper-type II_1.

Using Theorem 1 and the existence of ergodic hyperfinite subgraphs, we get:

Theorem 2. Every type III_0 ergodic mcp graph $\mathcal{G}$ admits an ergodic hyperfinite subgraph $\mathcal{H} \subseteq \mathcal{G}$ which is still III_0, in fact, has the same Krieger flow as $\mathcal{G}$.

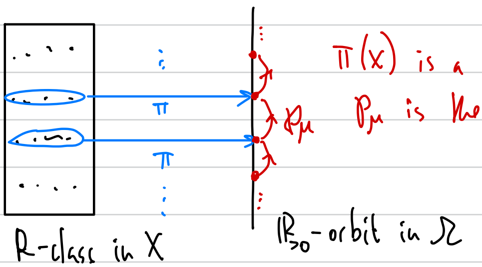
We then edge-slide a T_{p_0,q_0}-treeing along $\mathcal{H}$ into a T_{p_1,q_1}-treeing.

Krieger flow: Let R be an nsp ctbl Borel eq. rel. on st. prob. space (X, μ). Let $w : R \to \mathbb{R}_{>0}$ be the Radon–Nikodym cocycle.

The Krieger flow associated to R and $[\mu]_\sim$ is an nsp Borel action of $\mathbb{R}_{>0}$ on a st. prob. sp. (Ω, ν), which in particular has the following property:
$\exists$ $\ker(w)$-invariant Borel map $\pi : X \to \Omega$ such that

$$(y,x) \in E \implies \pi(y) = w(y,x)\,\pi(x).$$

Proposition R is type III_0 $\iff$ the Krieger flow is nowhere smooth and aperiodic.



$\pi(X)$ is a cross-section in Ω when w is equispaced.
P_μ is the Poincaré return map.

Theorem 0. Let R, μ_0 be as above and $\alpha_0, \alpha_1 \in \mathbb{R}_{>0}$. Suppose that the Radon–Nikodym cocycle is α_0-equispaced and the Poincaré return map φ_{μ_0} is pmp. If the Krieger flow admits an α_1-equispaced cross-section, then $\exists$ probability measure $\mu_1 \sim \mu_0$ whose Radon–Nikodym cocycle is α_1-equispaced and the Poincaré return map φ_{μ_1} is pmp.