Foliations, delocalized Chern characters and numeric ϱ-invariants

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Periodic Homology

A loc. convex algebra

- $\left(C_n(A) := \tilde{A} \otimes A^{\otimes n}, b, B\right) \qquad b^2 = B^2 = Bb + bB = 0$

$$BC_{p,q}(A) = \begin{cases} C_{q-p+1}(A) & q \geq p \\ 0 & \text{otherwise} \end{cases}$$

$$HP_*(A) := H_*\left(\mathrm{Tot}(BC(A)), b + B\right)$$

- $\alpha : A \to A' \rightsquigarrow \left(\underbrace{C_n(\alpha), b_\alpha}_{\text{mapping cone complex}}, \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & -B' \end{pmatrix}\right)$

$$HP_*(\alpha) := H_*\left(\mathrm{Tot}(BC(\alpha)), b_\alpha + \tilde{B}\right)$$

$\rightsquigarrow$ L.E.S. $\quad \cdots \to HP_{n+1}(A') \to HP_n(\alpha) \to HP_n(A) \to \cdots$

CHERN-CONNES CARACHTER

$$K_0(A) \xrightarrow{ch} HP_0(A)$$

$$\begin{array}{c} e \\ \cap \\ \lim\limits_{n\to+\infty} M_n(A) \end{array} \longmapsto \left(tr_0(e) + \sum_{k=1}^{\infty} (-2\pi i)^k \frac{2k}{k!} tr_{2k}\left(\left(e-\tfrac{1}{2}\right) \otimes e^{\otimes 2k} \right) \right)$$

(analogous formulas for the odd case)

Ex If $A = C^\infty(M)$ with M smooth closed manifold

$$\Rightarrow \quad \begin{array}{ccc} K_0(A) & \xrightarrow{ch} & HP_0(A) \\ \| & & \| \\ K^0(M) & & \bigoplus\limits_{k=0}^{\infty} H^{2k}_{dR}(M) \end{array}$$

we obtain the classical Chern carachter.

RELATIVE CHERN CHARACTERS

$$\begin{cases} p, q \text{ projections in } M_n(A) \\ h \text{ path of projections between } \alpha(p) \text{ and } \alpha(q) \text{ in } M_n(A') \end{cases}$$

$$\underset{\displaystyle HP_0(\alpha)}{\underset{\cap}{ch(p,q;h)}} = \Big(ch(q) - ch(p) \,,\, \underbrace{-\int_0^1 \cancel{ch}\big(h(s), 2(h(s)-1)\dot{h}(s)\big)\,ds}_{T\cancel{ch}(h)} \Big)$$

transgression formula

And we have a similar construction for the odd case in $HP_1(\alpha)$.

$HP_*(C_c^\infty(G))$ for G étale groupoid

$\boxed{\text{Rmk}}$ $C_c^\infty(G)^{\otimes n} \cong C_c^\infty(G^n)$

Def $G_\gamma^{(n)} := \{ (\gamma_0, \dots, \gamma_{n-1}) \in G^{(n)} \mid s(\gamma_0) = r(\gamma_n) \}$

Loops space or Burghelea space

Prop The restriction map $C_c^\infty(G^n) \to C_c^\infty(G_\gamma^{(n)})$ induces an isomorphism on HP_*

If $O \subset G_g^{(0)}$ is an open-closed subset invariant under conjugation

$\Rightarrow \chi_O$ is an idempotent on the complex $C_c^\infty(G_g^{(n)})$

characteristic function of O

Def $HP(C_c^\infty(G))_O :=$ homology of the image of χ_O

$\rightsquigarrow$ If $G_g^{(0)} = \sqcup O$ disjoint, open, G-inv. cover

$$\Rightarrow HP_*(C_c^\infty(G)) \cong \bigoplus HP_*(C_c^\infty(G))_O$$

NOTATION

- $HP_*^{[1]}(C_c^\infty(G)) := HP_*(C_c^\infty(G))_{G^{(0)}}$ ↖ unit space of G

  Localized homology

- $HP_*^{del}(C_c^\infty(G)) := HP_*(\iota)$ ↖ relative homology

  $\iota : \chi_{G^{(0)}} C_c^\infty(G_g^{(n)}) \hookrightarrow C_c^\infty(G_g^{(n)})$

  Delocalized homology

$\rightsquigarrow$ We obtain a long exact sequence

INDEX VIA DNC

$G \rightrightarrows M$ Lie groupoid

Def $G^{[0,1]}_{ad} := AG \times \{0\} \sqcup G \times]0,1] \rightrightarrows M \times [0,1]$

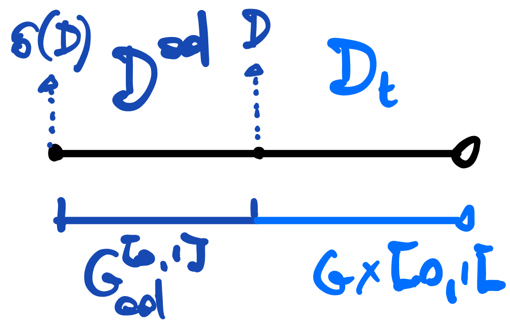
$$\begin{tikzcd} K_*(C^*_r(AG)) \arrow[r, dotted, shift right] & K_*(C^*_r(G^{[0,1]}_{ad})) \arrow[l, "ev_0"', shift right] \arrow[r, "ev_1"] & K_*(C^*_r(G)) \\ {[\sigma(D)]} & {[D^{ad}]} \arrow[l] \arrow[r] & {[D]} \end{tikzcd}$$

$\mathrm{Ind}_G : K_*(C^*_r(AG)) \longrightarrow K_*(C^*_r(G))$

Adiabatic L.E.S. and ϱ-classes

$$\cdots \to K_*(G\times]0,1[) \to K_*(G_{ad}^{[0,1[}) \xrightarrow{ev_0} K_*(AG) \xrightarrow{\Sigma\,\mathrm{Ind}_G} K_{*+1}(G\times]0,1[) \to$$

- If $[D]=0$ in $K_*(G)$



$D_0 = D$

D_1 invertible

Secondary invariant

$$\boxed{\varrho(D, D_t) \text{ in } K_*(G_{ad}^{[0,1[})}$$

Ex

$D = D_G^{sign}$

$D_t = D_G^{sign} + tC$

Hilsum-Skandalis

FOLIATIONS M closed mfd of dim n

Def A (regular) foliation of codim q on M is given by $\{\varphi_i : U_i \to L_i \times T_i = I^p \times I^q\}$ with $\varphi_{ij}(x,y) = (\lambda_{ij}(x,y), \tau_{ij}(y))$

The plaques $\varphi_i^{-1}(L_i \times \{t\})$ amalgamate to leaves and the sub-bundle $F \subset TM$ of vectors tangent to the leaves is an integrable Lie algebroid.

The monodromy groupoid of (M,F) is

$$\mathrm{Mon}(M,F) = \{\alpha : [0,1] \to M \mid \alpha([0,1]) \subset \text{leaf}\} / \sim$$

Examples

- $F = TM \rightsquigarrow \mathrm{Mon}(M,TM) = \tilde{M} \times_{\Gamma} \tilde{M}$ fundamental gpd

- $\Gamma \to \tilde{X} \to X$ and $\Gamma \curvearrowright B$ cpt
  universal covering

$$\left.\begin{array}{l} M = (\tilde{X} \times B)/\Gamma \\ F = \pi_* T\tilde{X} \end{array}\right\} \rightsquigarrow \mathrm{Mon}(M,F) = (\tilde{X} \times \tilde{X} \times B)/\Gamma$$

CM map

$G = \mathrm{Mon}(M,F)$

$\mathcal{U} = \{U_i \xrightarrow{\varphi_i} L_i \times T_i\}$ finite cover of M

① $G_{\mathcal{U}} = \bigsqcup_{i,j} G_{U_i}^{U_j} \rightrightarrows \bigsqcup_k U_k$

② $S_i = \varphi_i^{-1}(\{0\}\times T_i)$ $\rightsquigarrow$ $G_{\mathcal{U}} \overset{\varphi}{\cong} L_i \times L_j \times \Gamma_{ij}^S \circlearrowleft G_{S_i}^{S_j}$

local transversal

③ $\Gamma^S = \bigsqcup \Gamma_{ij}^S \rightrightarrows \bigsqcup S_k$ étale groupoid

Def $\Phi^S : C_c^\infty(G) \longrightarrow C_c^\infty(\Gamma^S)$

$$\xi \mapsto (\alpha_i^{1/2} \xi \alpha_j^{1/2})_{ij} \overset{\varphi^*}{\mapsto} (\xi_{ij}^S \otimes k_{ij})_{ij} \overset{Tr}{\mapsto} \sum_i Tr(k_{ii})\, \xi_{ii}^S$$

$\{\alpha_i\}$ p.u.

Ex. If $F = TM \Rightarrow x \in M$ is a complete transversal

$$\Phi^{\{x\}} : HP_*(C_c^\infty(G)) \to HP_*(\mathbb{C}\pi_1(M,x))$$

$f^0 \otimes \dots \otimes f^n$ is sent to the function that to $(\gamma_0, \dots, \gamma_n)$ associates

$$\sum_{i_0,\dots,i_n} \left(\int_{M^{n+1}} \alpha_{i_0}^{\frac{1}{2}} f^0 \alpha_{i_1}^{\frac{1}{2}} ([\beta_{i_0}(x_0), \gamma_0 \beta_{i_1}(x_1)]) \dots \alpha_{i_n}^{\frac{1}{2}} f^n \alpha_{i_0}^{\frac{1}{2}} ([\beta_{i_n}(x_n), \gamma_n \beta_{i_0}(x_1)]) \right)$$

where $\beta_i : U_i \to \tilde{M}$ are smooth cross-sections of the projection $\hat{M} \to M$.

$\boxed{\text{Th}}$ (Crainic-Moerdijk)

Φ^S induces an isomorphism on HP.

Set $\Phi^S_t := \Phi^S \circ m_t : C_c^\infty((G_{ad}^{[0,1[})^n) \longrightarrow C_c^\infty((\Gamma^S)^n)$

$\boxed{\text{Prop}}$ $\Phi^S_0 := \lim_{t\to 0} \Phi^S_t$ induces a well-defined map $C_c^\infty(AG_{\mathfrak{z}}^{(n)}) \to C_c^\infty(((\Gamma^S)^{(n)})_{G^{(0)}})$

"Proof" compact subsets of $G_{ad}^{[0,1[}$ concentrate around $G^{(0)}$ as $t \to 0$.

CHERN CHARACTERS

- Fix $\mathcal{A}(\Gamma^S)$ dense hol. closed $\subset C^*_r(\Gamma^S)$
- Obtain $\mathcal{A}(G)$ dense hol. closed $\subset C^*_r(G)$ by "pulling-back" through the CM-map.
- Put $\mathcal{A}(G_{ad}^{[0,1)}) = \mathcal{S}(G_{ad}^{[0,1)}) + \mathcal{A}(G)(0,1)$ Schwartz functions

  is dense hol. closed by a Theorem of Lauter-Monthubert-Nistor.

- Ξ^S_t extends by construction.

Def

- $\underline{Ch_s^{[1]}} : K_*(\mathcal{S}(AG)) \longrightarrow HP_*^{[1]}(\mathcal{A}(\Gamma^S))$

  localized at [1]

$$\Phi_0^S \circ ch_*$$

- $\underline{Ch_s} : K_*(\mathcal{A}(G)[0,1]) \longrightarrow HP_*(\mathcal{A}(\Gamma^S))$

$$[p_t] \longmapsto \int_0^1 \Phi_t^S\left(\cancel{ch}(p_t, 2(p_t-1)\dot{p}_t)\right)dt$$

- $\underline{Ch_S^{del}} : K_*(\mathcal{A}(G_{ad}^{[0,1[})) \longrightarrow HP_*^{del}(\mathcal{A}(\Gamma^S))$

  delocalized

$$[p_t] \longmapsto \left(Ch_S^{[1]}(p_0), \int_0^1 \Phi_t^S\left(\cancel{ch}(p_t, 2(p_t-1)\dot{p}_t)\right)dt\right)$$

Th The following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & K_*(\mathcal{A}(G)(0,1)) \arrow[r, "\iota_*"] \arrow[d, "Ch_S"'] & K_*(\mathcal{A}(G^{[0,1[}_{ad})) \arrow[r, "ev_0"] \arrow[d, "Ch_S^{del}"'] & K_*(\mathcal{S}(AG)) \arrow[r, "\partial"] \arrow[d, "Ch_S^{[1]}"'] & \cdots \\
\cdots \arrow[r] & HP_{*+1}(\mathcal{A}(\Gamma^S)) \arrow[r] & HP_*^{del}(\mathcal{A}(\Gamma^S)) \arrow[r] & HP_*^{[1]}(\mathcal{A}(\Gamma^S)) \arrow[r] & \cdots
\end{tikzcd}$$

is commutative.

FUNCTORIALITY

$$G^{\varphi}_{\varphi} \rightrightarrows M_1 \xrightarrow{\varphi} M_2 \leftleftarrows G \qquad \varphi \text{ transverse w.r.t. } G$$

$\boxed{\text{Th}}$ The following diagram is commutative

$$\begin{tikzcd}
\cdots \arrow[r] & K_*(G^{\varphi}_{\varphi}\times(0,1)) \arrow[r] \arrow[d, "\mathrm{Mor}\,\varphi"] \arrow[dd, bend right, "Ch"'] & K_*\big((G^{\varphi}_{\varphi})^{[0,1[}_{ad}\big) \arrow[r] \arrow[d, "\varphi^{ad}_!"] \arrow[dd, bend right] & K_*(AG^{\varphi}_{\varphi}) \arrow[r] \arrow[d, "d\varphi_!"] \arrow[dd, bend right] & \cdots \\
\cdots \arrow[r] & K_*(G\times(0,1)) \arrow[r] \arrow[d] & K_*(G^{[0,1[}_{ad}) \arrow[r] \arrow[d] & K_*(AG) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & HP_{*+1}(\mathbb{Q}(\Gamma^s)) \arrow[r] & HP^{ad}_*(\mathbb{Q}(\Gamma^s)) \arrow[r] & HP^{[I]}_*(\mathbb{Q}(\Gamma^s)) \arrow[r] & {}
\end{tikzcd}$$

Applications:

a) $(M_1, F_1) \xrightarrow{h} (M_2, F_2)$ foliated homotopy equivalence

b) g_F is a longitudinal psc metric on (M, F).

We can then define:

Hilsum-Skandalis

a) $Ch^{del}_{\Gamma_2^s}(\varrho(D^{sign}_{F_1 \cup F_2} + C_h))$

b) $Ch^{del}_{\Gamma^s}(\varrho(\not{D}_{g_F}))$

• (M, F) with Γ^S associated étale grpd

$M = \partial W$ and $W \longrightarrow B\Gamma^S$ ← Haefliger classifying sp.

s.t. the induced foliation on W is transverse to ∂W and restricts to F.

Homological Deloc. APS index thm for foliations

$$\iota_* Ch_S(\mathrm{Ind}(D_W)) = Ch_S^{del}(\varrho(D_M + C))$$

invertible

Thank you!